Exceptional collections in algebraic geometry

LMU
Fall 2024

Introduction

Nonlinear objects ⇝ linear invariants:

X - top. space ⇝ $H_*(X)$, $H^*(X)$ - homology & cohomology

G - group ⇝ representations of G

R - algebra ⇝ modules over R

X - alg. variety ⇝ sheaves on X

X, Y - top. spaces. X is hom. equiv. to $Y \Rightarrow H_*(X) \simeq H_*(Y)$, $H^*(X) \simeq H^*(Y)$

Converse is not true: e.g. $\underset{\text{reduced suspension}}{\Sigma} \mathbb{CP}^2$ & $S^3 \vee S^5$ $\quad H_*, H^* \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

moreover, $H^*(X) \simeq H^*(Y)$ as rings (but not over Steenrod alg.) --- Massey product, etc.

Recall that $H_i(X) := H_i(\underset{\text{singular compex of } X\text{, i.e.}}{C_\bullet(X)})$

$$\begin{tikzcd} \cdots \arrow[r] & C_2(X) \arrow[r, "d_1"] \arrow[d, equal] & C_1(X) \arrow[r, "d_0"] \arrow[d, equal] & C_0(X) \arrow[d, equal] \\ & \bigoplus\limits_{\delta: \Delta^2 \to X} \mathbb{Z} & \bigoplus\limits_{\delta: [0,1] \to X} \mathbb{Z} & \bigoplus\limits_{x \in X} \mathbb{Z} \end{tikzcd}$$

Here one has $d_i \circ d_{i+1} = 0$ and $H_i(C_\bullet(X)) := {}^{\ker d_{i-1}}\!/_{\operatorname{Im} d_i}$

One can endow X with additional structure, e.g. triangulation or cellular structure ⇝ $C^{tr}_\bullet(X)$ & $C^{cell}_\bullet(X)$,

$C^{tr}_i(X) := \bigoplus\limits_{\Delta^i \text{ in triang.}} \mathbb{Z} \qquad C^{cell}_i(X) := \bigoplus\limits_{\substack{\text{cells in } X \\ \text{of dim. } i}} \mathbb{Z}$

One has $H_i(C_\bullet(X)) \simeq H_i(C^{tr}_\bullet(X)) \simeq H_i(C^{cell}_\bullet(X))$.

Th (Whitehead) X, Y - (nice) top. spaces, $\pi_1(X) = \pi_1(Y) = *$. Then

X is hom. equiv. to $Y \Longleftrightarrow \exists\ \delta: C_\bullet(X) \to C_\bullet(Y)$ s.t. δ induces isoms on $H_i(-)$.
(i.e. δ - quasi-iso-sm of complexes).

~~($\Leftrightarrow \exists\ \tilde{C}$ and $\delta_1: \tilde{C} \to C^{tr}_\bullet(X)$, $\delta_2: \tilde{C} \to C^{tr}_\bullet(Y)$ - qis) $\Leftrightarrow$ same for $C^{cell}_\bullet(-)$~~

$\rightsquigarrow C_\bullet(X)$ $\overset{(\& \, C^{tr}_\bullet(X),\, C^{cell}_\bullet(X))}{}$ carries more information then $H_i(X)$, in particular, it remembers X up to hom. equiv (for simply connected spaces)

Recall also that $H^i(X) := H^i(\underline{Hom}(C_\bullet(X), \mathbb{Z}))$

dual complex:

$$Hom(C_0(X), \mathbb{Z}) \xrightarrow{d_0^\vee} Hom(C_1(X), \mathbb{Z}) \xrightarrow{d_1^\vee} Hom(C_2(X), \mathbb{Z}) \to \dots$$

and in general $H^i(\underline{Hom}(C_\bullet(X), \mathbb{Z})) \neq Hom(H_i(X), \mathbb{Z})$, e.g. if $H^i(X)$ has torsion : $H^2(\mathbb{RP}^2) = \mathbb{Z}/2$, $H_2(\mathbb{RP}^2) = 0$.

$\rightsquigarrow$ one takes dual of the complex, not homology (complexes carry more information)

G - group $\rightsquigarrow$ representations of $G \xleftrightarrow{1:1} \mathbb{C}[G]$-modules

R - $\mathbb{C}$-algebra $\rightsquigarrow$ R-modules

$\Big\} \longrightarrow D(R)$ - derived category of R-modules, $D(R) = Kom(R)[qis^{-1}]$

If M, N - R-modules, how to classify extensions, i.e. exact sequences $0 \to N \to L \to M \to 0$. For example, are there such L s.t. $L \not\simeq N \oplus M$?

$$\begin{tikzcd} & 0 \to N \to L \to 0 \arrow[dl, "qis"'] \arrow[dr] & \\ \underset{-2}{0} \to \underset{-1}{0} \to \underset{0}{M} \to \underset{1}{0} & & 0 \to N \to 0 \to 0 \end{tikzcd}$$

$$\Big(\dots \Big) \in Hom_{D(R)}\big(\underset{-1}{0} \to \underset{0}{M} \to \underset{1}{0} \, , \, \underset{-2}{0} \to \underset{-1}{N} \to \underset{0}{0} \big) \xleftrightarrow{1:1} Ext(M, N)$$

$D(R)$ is also a home for $\underline{\text{tilting theory}}$ (i.e. some equivalences between (sub)categories of R_1-mod & R_2-mod for some R_1, R_2)

X-alg. variety $\rightsquigarrow D(X)$ - derived category of coherent sheaves.

Sometimes $D(X) \simeq D(X')$ but $X \not\simeq X'$ (e.g. $X = A$ - abelian variety, $X' = A^\vee$)

Moreover, sometimes $D(X) \simeq D(R)$ for an algebra R, e.g.

$$D(\mathbb{P}^1) \simeq D(\mathbb{C}\langle \bullet \rightrightarrows \bullet \rangle)$$

quiver algebra, $= \mathbb{C}\langle e_1, e_2, X_{12}^{(1)}, X_{12}^{(2)} \rangle / e_1^2 = e_1$, $e_2^2 = e_2$, $X_{12}^{(1)} e_1 = X_{12}^{(1)}$, $e_2 X_{12}^{(1)} = X_{12}^{(1)}$, $X_{12}^{(2)} e_1 = X_{12}^{(2)} = e_2 X_{12}^{(2)}$, all other products $= 0$.

$\rightsquigarrow$ geometry of $\mathbb{P}^1$ is somehow related to repr. theory of the quiver (of the quier algebra), which is straight forward: module over $\mathbb{C}\langle \bullet \rightrightarrows \bullet \rangle$ $\longleftrightarrow V_1, V_2$ - vector spaces & $A_1, A_2 \in Hom(V_1, V_2)$.

M-module $\rightsquigarrow 0 \to \underset{V_2}{e_2 M} \to M \to \underset{V_1}{N} \to 0$ $\rightsquigarrow$ something similar for coherent modules over $\mathbb{P}^1$

The vertices of quiver correspond to $(\mathcal{O}, \mathcal{O}(1))$ – $\underline{\text{full exceptional collection}}$ on $\mathbb{P}^1$ – they allow to decompose $D(\mathbb{P}^1)$ into two "simple" parts $\simeq D(\mathbb{C})$ (Beilinson '78)

Similar holds for $\mathbb{P}^n$, $Gr(n,m)$, smooth proj. quadrics $/\mathbb{C}$, ...

Homological mirror symmetry (Kontsevich '94)
ICM talk

type II ~~super~~string theory $\begin{smallmatrix}\nearrow \text{type IIA} \\ \searrow \text{type IIB}\end{smallmatrix}$ $\rightsquigarrow$ "mirror symmetric" Calabi-Yau manifolds

CY manifold: compact Kähler manifold s.t. canonical bundle is trivial

manifold with compatible complex, symplectic & Riemannian structures, i.e. complex manifold with hermitian metric h such that the associated 2-form $\operatorname{Im} h(-,-) \in \Lambda^2 T_M^*$ is closed

~~In particular, enumeration~~ complex-algebraic

Mirror symmetry flips ~~algebraic~~ & symmetric sides.

In particular, enumeration of alg. gadgets on CY manifold (e.g. count of rational curves)

$\leftrightarrow$ enumeration of symplectic gadgets on the mirror CY manifold.

Kontsevich: HMS: $D(X) \simeq$ Fukaya category of the mirror CY-manifold.

Rough plan:
1) Derived cats & some properties
2) recollection on alg. geometry
3) reconstruction theorems: when $D(X)$ remembers X?
4) exceptional collections on some varieties.

Derived category of modules

R - associative $\mathbb{C}$-algebra with 1.

R-Mod - category of left R-modules

Kom(R) - category of complexes of R-Mod, i.e.

Objects: $M^\bullet = \ldots \to M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \to \ldots$ s.t. $d^n \circ d^{n-1} = 0$ $\forall n \in \mathbb{Z}$

Morphisms: $f: M^\bullet \to N^\bullet$ is a sequence $\{f^n\}^{n \in \mathbb{Z}}$ s.t.

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[r] & M^n \arrow[r, "d_M^n"] \arrow[d, "f^n"'] & M^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & N^n \arrow[r, "d_N^n"] & N^{n+1} \arrow[r] & {}
\end{tikzcd}$$

commutes

There is a canonical functor $R\text{-Mod} \to \mathrm{Kom}(R)$

$$M \longmapsto M^\bullet \text{ s.t. } M^n = \begin{cases} M, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

There is a shift functor $[1]: Kom(R) \to Kom(R)$

$$\begin{tikzcd} \to M^{n-1} \arrow[r, "d^{n-1}"] & M^n \arrow[r, "d^n"] \arrow[d, phantom, "\wr"] & M^{n+1} \to \\ \to M^n \arrow[r, "-d^n"] & M^{n+1} \arrow[r, "-d^{n+1}"] & M^{n+2} \to \end{tikzcd}$$

$M^\circ \mapsto (M[1])^\circ$ s.t. $M[1]^n = M^{n+1}$

$f: M^\circ \to N^\circ \mapsto f[1]$, $f[1]^n := f^{n+1}$ $\quad d^n_{M[1]} = -d^{n+1}_M$

$[1]$ is an equivalence of categories.

$M^\circ \in Kom(R)$, $n \in \mathbb{Z} \rightsquigarrow H^n(M^\circ) := \ker d^n / \mathrm{Im}\, d^{n-1} \in R\text{-Mod}$

$\nwarrow$ cohomology of M°

$f \in Hom_{Kom(R)}(M^\circ, N^\circ) \rightsquigarrow H^n(f): H^n(M^\circ) \to H^n(N^\circ)$.

Morphism $f: M^\circ \to N^\circ$ is a quasi-isomorphism (qis) if $H^n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$

Def. Functor $F: Kom(R) \to \mathcal{C}$ inverts qis-s if $\forall M^\circ, N^\circ \in Kom(R)$ and $\forall f: M^\circ \to N^\circ$ - qis morphism $F(f)$ is an isomorphism.

Thm. $\exists!$ universal initial functor $Q: Kom(R) \to D(R)$ inverting qis-s, i.e. 1) Q inverts qis-s

2) $\forall F: Kom(R) \to \mathcal{C}$ inverting qis-s $\exists$ unique $G: D(R) \to \mathcal{C}$ s.t. $F = G \circ Q$, i.e.

$$\begin{tikzcd} Kom(R) \arrow[r, "Q"] \arrow[dr, "F"'] & D(R) \arrow[d, "G"] \\ & \mathcal{C} \end{tikzcd}$$

commutes

Rk this is an example of a localization of a category.

The definition is not really good (not stable under equivalences of cats)

Better: $\forall F\ \exists G$ st. $F \simeq G \circ Q$ & $\forall G_1, G_2: D(R) \to \mathcal{C}$ the induced $Hom(G_1, G_2) \xrightarrow{\circ Q} Hom(G_1 \circ Q, G_2 \circ Q)$ is a bijection.

$\nearrow$ isom-sm of functors

Pf $Ob(D(R)) = Ob(Kom(R))$,

$Hom_{D(R)}(M^\circ, N^\circ) =$

$$= \left\{ \begin{tikzcd}[column sep=small] & P_1^\circ \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & P_2^\circ \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & & & P_n^\circ \arrow[dr, "f_n"] & \\ M^\circ & & L_1^\circ & & L_2 \cdots & & & N^\circ \end{tikzcd} \right\} \Big/ \text{substitutions}$$

where s_i - qis $\nearrow\!\!\swarrow$

"$f_n s_n^{-1} f_{n-1} \dots f_1 s_1^{-1}$"

- $P_i \rightsquigarrow \begin{tikzcd}[column sep=small] P_i \arrow[dr, "s"] & & P_i \arrow[dl, "s"'] \\ & L & \end{tikzcd}$ $\forall s$ - qis
- $L_i \rightsquigarrow \begin{tikzcd}[column sep=small] & P \arrow[dl, "s"'] \arrow[dr, "s"] & \\ L_i & & L_i \end{tikzcd}$
- $\begin{tikzcd}[column sep=small] P_i \arrow[dr, "f"] & & L_i \arrow[dl, "id"'] \arrow[dr, "g"] & \\ & L_i & & L_{i+1} \end{tikzcd} \rightsquigarrow \begin{tikzcd} P_i \arrow[dr, "gf"] & \\ & L_{i+1} \end{tikzcd}$

$id_{M^\bullet} = \{ M^\bullet \}$

composition = concatenation

Note that every morphism is a composition of $\begin{tikzcd}[column sep=small] & L^\bullet \arrow[dl, "s"'] \arrow[dr, "id"] & \\ M^\bullet & & L^\bullet \end{tikzcd}$ and

$$\begin{tikzcd}[column sep=small] & L^\bullet \arrow[dl, "id"'] \arrow[dr, "f"] & \\ L^\bullet & & N^\bullet \end{tikzcd}$$

Functor $Q : Kom(R) \to D(R)$

$$M^\bullet \mapsto M^\bullet$$

$$f : M^\bullet \to N^\bullet \mapsto \begin{tikzcd}[column sep=small] & M^\bullet \arrow[dl, "id"'] \arrow[dr, "f"] & \\ M^\bullet & & N^\bullet \end{tikzcd}$$

$Q(s)^{-1} = \begin{tikzcd}[column sep=small] & N \arrow[dl, "s"'] \arrow[dr, "id"] & \\ M^\bullet & & N^\bullet \end{tikzcd}$ $\forall s$ - qis $\rightsquigarrow$ Q inverts qis.

$G(M^\bullet) := F(M^\bullet)$

$G(f_n \dots f_2 s_2^{-1} f_1 s_1^{-1}) := F(f_n) \dots F(s_2)^{-1} F(f_1) F(s_1^{*})^{-1}$ <u>Exercise</u>: correctly defined.

<u>Rk</u> 1) size-theoretic issues: $Hom_{D(R)}(M^\bullet, N^\bullet)$ may fail to be a set (can be a proper class)

$\rightsquigarrow$ one may consider only modules of bounded cardinality

2) How to compute $Hom_{D(R)}(M^\bullet, N^\bullet)$?

3) It follows from the universal property that $\forall i \in \mathbb{Z}$ $H^i(-) : Kom(R) \to R\text{-Mod}$ extends to $D(R)$

<u>Def.</u> Class of morphisms S in a category $\mathcal{C}$ is <u>localizing</u> if

1) $\forall X \in \mathcal{C}$ $id_X \in S$
$\forall s, t \in S$ $s \circ t \in S$
(if defined)

2) $\begin{tikzcd}[column sep=small] & P \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M \arrow[dr, dashed, "g"'] & \circlearrowleft & N \arrow[dl, dashed, "t"] \\ & L & \end{tikzcd}$ $\begin{tikzcd}[column sep=small] & P \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "g"] & \\ M \arrow[dr, "f"'] & \circlearrowleft & N \arrow[dl, "s"] \\ & L & \end{tikzcd}$ $\forall f \in \mathcal{C}, s \in S$ $\exists g \in \mathcal{C}, t \in S$

3) $f, g : M \to N \Rightarrow (\exists s \in S$ s.t. $sf = sg \Leftrightarrow \exists t \in S$ s.g $ft = gt)$.

<u>Rk</u> $\{qis\} \subseteq Kom(R)$ in general <u>not</u> localizing

<u>Def.</u> $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ is a <u>short exact sequence</u> of R-Modules is f is inj,
g is surj, $Im f = ker g$.

A s.e.s. splits if there exists $s : L \to N$ s.t. $g \circ s = id_L$ <u>Exercise</u>: s.e.s. splits $\Rightarrow N \simeq M \oplus L$

<u>Ex</u>: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/_2 \to 0$ does <u>not</u> split

<u>Prop.</u> Suppose that in R-Mod all s.e.s. split (e.g. R is a skew-field)

$\Rightarrow D(R) \simeq Kom_0(R) \simeq \prod_{i \in \mathbb{Z}} R\text{-Mod}$, where $Kom_0(R) \subseteq Kom(R)$ is the full subcategory of complexes with all $d = 0$.

Pf. Let $M^\bullet \in \mathrm{Kom}(R)$, put $B^n := \mathrm{Im}\, d^{n-1} \subseteq M^n$, $Z^n := \ker d^n \subseteq M^n$, $H^n := H^n(M^\bullet)$ $\rightsquigarrow$

$$\left.\begin{array}{l} 0 \to B^n \to Z^n \to H^n \to 0 \\ 0 \to Z^n \to M^n \to B^{n+1} \to 0 \end{array}\right\} \text{- s.e.s.} \overset{\text{splitting}}{\Longrightarrow} M^n \simeq Z^n \oplus B^{n+1} \simeq B^n \oplus B^{n+1} \oplus H^n$$

Furthermore, $d^n : M^n \to M^{n+1}$ is given by

$$\begin{tikzcd} M^n \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] & M^{n+1} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\ B^n \oplus B^{n+1} \oplus H^n & B^{n+1} \oplus B^{n+2} \oplus H^{n+1} \end{tikzcd} \quad (b_n, b_{n+1}, h_n) \mapsto (b_{n+1}, 0, 0).$$

Let $H^\bullet(M^\bullet) \in \mathrm{Kom}_0(R)$ be $\left(\ldots \xrightarrow{0} H^n(M^\bullet) \xrightarrow{0} H^{n+1}(M^\bullet) \xrightarrow{0} \right)$

$\rightsquigarrow$ $f : M^\bullet \to H^\bullet(M^\bullet)$, $f^n : M^n \to H^n(M^\bullet)$
$(b_n, b_{n+1}, h_n) \mapsto h_n$

$g : H^\bullet(M^\bullet) \to M^\bullet$, $g^n : H^n(M^\bullet) \to M^n$
$h_n \mapsto (0, 0, h_n)$.

f, g are identity on cohomology $\Rightarrow$ qi's

$$\begin{tikzcd} \mathrm{Kom}(R) \arrow[r, "Q"] \arrow[d, shift right, "F"'] & D(R) \arrow[dl, dotted, "G"] \\ \mathrm{Kom}_0(R) \arrow[u, shift right, "i"'] & \end{tikzcd} \qquad M^\bullet \mapsto H^\bullet(M^\bullet)$$

$F \circ i = \mathrm{id} \Rightarrow G \circ Q \circ i = \mathrm{id}_{\mathrm{Kom}_0(R)}$

$(Q \circ i \circ G)(M^\bullet) = H^\bullet(M^\bullet)$

$\rightsquigarrow$ $Q \circ i \circ G \underset{\substack{\text{isomorphism} \\ \text{of functors}}}{\simeq} \mathrm{id}_{D(R)}$ via f, g.

Def. $f, g : M^\bullet \to N^\bullet$ are <u>homotopically equivalent</u> (write $f \sim g$) if $\exists \{h^n : M^n \to N^{n-1}\}_{n \in \mathbb{Z}}$ s.t. $f^n - g^n = h^{n+1} d^n_{M^\bullet} + d^{n-1}_{N^\bullet} h^n$

$$\begin{tikzcd} \ldots & M^{n-1} \arrow[r] \arrow[d, "f^{n-1} - g^{n-1}"] & M^n \arrow[r, "d"] \arrow[d, "f^n - g^n"] \arrow[dl, "h^n"'] & M^{n+1} \arrow[d, "f^{n+1} - g^{n+1}"] \arrow[dl, "h^{n+1}"'] & \ldots \\ \ldots & N^{n-1} \arrow[r, "d"'] & N^n \arrow[r] & N^{n+1} & \ldots \end{tikzcd}$$

Rk. 1) $C_i^{cell}(X \times [0,1]) = C_i^{cell}(X) \oplus C_i^{cell}(X) \oplus C_{i-1}^{cell}(X)$

cells: $U_i \times \{0\}, U_i \times \{1\}, U_i \times (0,1)$

homotopy $H : X \times [0,1] \to Y$ yields maps

$$\begin{tikzcd} C_i^{cell}(X) \oplus C_i^{cell}(X) \oplus C_{i-1}^{cell}(X) \arrow[r, "h"] \arrow[r, bend left, "\text{\`\`}H|_{X \times \{0\}}\text{''}"] \arrow[r, bend right, "\text{\`\`}H|_{X \times \{1\}}\text{''}"'] & C_i(Y) \end{tikzcd}$$

2) Cohomological / homological notation:

$$\ldots \to M^n \to M^{n+1} \to \ldots \quad \leftrightsquigarrow \quad \ldots \to \underset{\substack{\| \\ M^{-n-1}}}{M_{n+1}} \to \underset{\substack{\| \\ M^{-n}}}{M_n} \to \ldots$$

Exercise: If $F, G : X \to Y$ - hom. equiv. maps of CW-complexes the the induced $f, g : C_\bullet^{cell}(X) \to C_\bullet^{cell}(Y)$ - hom. equiv.
(if you know alg. top.)

Lm. 1) hom. equiv. is an equiv. relation

2) $f_1, f_2, g_1, g_2 : M^\circ \to N^\circ$, $f_1 \sim g_1$, $f_2 \sim g_2 \Rightarrow f_1 + f_2 \sim g_1 + g_2$.

3) $f \sim g \Rightarrow f \circ \alpha \sim g \circ \alpha$, $\beta \circ f \sim \beta \circ g$ $\forall \alpha, \beta$.

4) $f, g : M^\circ \to N^\circ$, $f \sim g \Rightarrow H^n(f) = H^n(g) : H^n(M^\circ) \to H^n(N^\circ)$ $\forall n \in \mathbb{Z}$.

Pf 1) $f \sim g \Rightarrow g \sim f$ with $-h$; $f \sim f$ for $h = 0$; $f \sim g$ with h, $g \sim r$ with $\tilde{h}$ $\Rightarrow f \sim r$ with $h + \tilde{h}$

2) via $h_1 + h_2$

3) $f\alpha - g\alpha = hd\alpha + dh\alpha = (h\alpha)d + dh\alpha$, i.e. homotopy given by $h\alpha$.

$\beta f \sim \beta g$ via βh.

4) suffices to check that for $f \sim 0$ $H^n(f) = 0$.

$a \in H^n(M^\circ) \rightsquigarrow \bar{a} \in M^n$ representing $a \rightsquigarrow f(\bar{a}) = h\underbrace{d\bar{a}}_{0, \bar{a} \in \ker d_M} + \underbrace{dh\bar{a}}_{\in \operatorname{Im} d_N} \in \operatorname{Im} d_N$

Def. $K(R)$ – homotopy category of R:

$\operatorname{Ob} K(R) := \operatorname{Ob} Kom(R)$

$\operatorname{Hom}_{K(R)}(M^\circ, N^\circ) := \operatorname{Hom}_{Kom(R)}(M^\circ, N^\circ) / \sim$

Rk. 1) $\forall n \in \mathbb{Z}$ $H^n(-) : Kom(R) \to R\text{-Mod}$ descends to $K(R) \to R\text{-Mod}$.

2) $f \sim g$, ~~f~~ f-qis $\Rightarrow g$-qis $\Rightarrow$ one has notion of qis in $K(R)$; qis form a localizing class in $K(R)$ (we will see it).

Def. $f \in \operatorname{Hom}_{Kom(R)}(M^\circ, N^\circ) \rightsquigarrow$ Cone of f is the complex $C(f)$ with

$C(f)^n := M^{n+1} \oplus N^n$

$d^n_{C(f)} : M^{n+1} \oplus N^n \to M^{n+2} \oplus N^{n+1}$, $(a, b) \mapsto (-d^{n+1}_M(a), f^{n+1}(a) + d^n_N(b))$

$\begin{pmatrix} d_{M[1]} & 0 \\ f[1] & d_N \end{pmatrix}$ [illegible]

Rk. $F : X \to Y$ $\rightsquigarrow$  $:= X \times [0,1] \amalg Y / (x,0) \sim (x',0),\ (x,1) \sim f(x)$

Ex: $f : M \to N$ – homo-sm of R-Mod. Then, viewing f as a morphism of complexes concentrated in degree 0, $\operatorname{Cone}(f) = (\ldots 0 \to \underset{-1}{M} \xrightarrow{f} \underset{0}{N} \to 0 \to \ldots)$

Def. $_i$ -**exact** if $\forall i$ $\operatorname{Im}\delta_n = \ker\delta_{n+1}$.

$$\to P^n_\bullet \xrightarrow{\delta_n} P^{n+1} \to \dots$$

Thm. $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$ - degreewise s.e.s. of complexes $\Rightarrow$ There is a long exact sequence

$$\dots \to H^n(A^\bullet) \xrightarrow{H^n(f)} H^n(B^\bullet) \xrightarrow{H^n(g)} H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(A^\bullet) \to \dots$$

Pf.

$$\begin{tikzcd}
& \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \\
0 \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & B^{n-1} \arrow[r] \arrow[d] & C^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^n \arrow[r] \arrow[d] & B^n \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r] \arrow[d] & B^{n+1} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & 0 \\
& \dots & \dots & \dots &
\end{tikzcd}$$

$c \in H^n(C^\bullet) \rightsquigarrow$ representative $\bar{c} \in C^n$

s.t. $d\bar{c} = 0 \rightsquigarrow b \in B^n$ s.t. $g(b) = \bar{c}$

$\Rightarrow g\,d\,b = d\,g\,b = d\bar{c} = 0 \rightsquigarrow$

$\bar{a} \in A^{n+1}$ s.t. $f(\bar{a}) = db$

$f\,d\bar{a} = d f\bar{a} = d\,d\,b = 0 \Rightarrow d\bar{a} = 0$

$\rightsquigarrow a \in \ker d^{n+1}_A / \operatorname{Im} d^n_A = H^{n+1}(A^\bullet)$.

$\delta(c) := a$

Exercise: 1) does not depend on choices

2) check exactness

Corollary $f \in \operatorname{Hom}_{\operatorname{Kom}(R)}(M^\bullet, N^\bullet) \Rightarrow$ there is a long exact sequence.

$$\dots \to H^n(M^\bullet) \xrightarrow{H^n(f)} H^n(N^\bullet) \xrightarrow{H^0(\tau)} H^n(C(f)) \xrightarrow{H^0(\pi)} H^n(M[1]) \simeq H^{n+1}(M^\bullet) \to \dots$$

where $N^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} M[1]$

$N^n \ni a \mapsto (0, a) \in M^{n+1} \oplus N^n$

$(a, b) \mapsto a$

Pf. $N^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} M[1]$ - degreewise s.e.s. of complexes $\Rightarrow$ long exact sequence.

$$\dots \to H^n(N^\bullet) \xrightarrow{H^n(\tau)} H^n(C(f)) \xrightarrow{H^n(\pi)} H^n(M[1]) = H^{n+1}(M^\bullet) \xrightarrow{\delta} H^{n+1}(N^\bullet)$$

$\bar{c} \in M^{n+1} \rightsquigarrow (\bar{c}, 0) \rightsquigarrow d(\bar{c}, 0) = (-d(\bar{c}), f(\bar{c})) = (0, f(\bar{c}))$

$\rightsquigarrow f(\bar{c}) \in N^{n+1}$ satisfies $\tau f(\bar{c}) = d(\bar{c}, 0) \Rightarrow \delta(c) =$ class of $f(\bar{c})$ in $H^{n+1}(N^\bullet) = \ker d^{n+1}_N / \operatorname{Im} d^n_N$.

$\square$

Lm

$$\begin{tikzcd} M_1^\circ \arrow[r, "\delta_1"] \arrow[d, "u"'] & N_1^\circ \arrow[r, "\tau_1"] \arrow[d, "v"'] & C(\delta_1) \arrow[r, "\pi_1"] \arrow[d, dashed, "\exists w"] & M_1[1] \arrow[d, "u[1]"] \\ M_2^\circ \arrow[r, "\delta_2"] & N_2^\circ \arrow[r, "\tau_2"] & C(\delta_2) \arrow[r, "\pi_2"] & M_2[1] \end{tikzcd}$$

Pf. $w(\underset{M_1^{n+1}}{\overset{\cap}{a}}, \underset{N_1^n}{\overset{\cap}{b}}) := (u(a), v(b))$ Exercise: check commutativity.

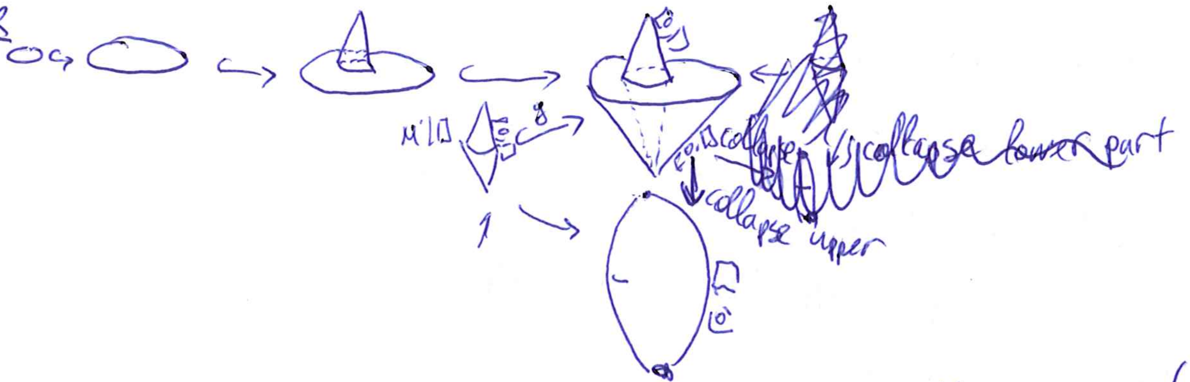
Lm $f \in \mathrm{Hom}_{\mathrm{Kom}(R)}(M^\circ, N^\circ) \Rightarrow \exists g$ - morphism of complexes s.t.

$$\begin{tikzcd} N^\circ \arrow[r, "\tau"] \arrow[d, "="'] & C(f) \arrow[r, "\pi"] \arrow[d, "="'] \arrow[dr, phantom, "①"] & M^\circ[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] \arrow[dr, phantom, "②"] & N^\circ[1] \arrow[d, "="] \\ N^\circ \arrow[r, "\tau"] & C(f) \arrow[r, "\overline{\tau}"] & C(\tau) \arrow[r, "\overline{\pi}"] & N^\circ[1] \end{tikzcd}$$

1) diagram commutes in $K(R)$

2) g is an isomorphism in $K(R)$

Rk



Pf. $C(\tau)^n = N^{n+1} \oplus C(f)^n = N^{n+1} \oplus M^{n+1} \oplus N^n$, $d = \begin{pmatrix} d_{N[1]} & \\ \tau[1] & d_{C(f)} \end{pmatrix} = \begin{pmatrix} d_{N[1]} & 0 & 0 \\ 0 & d_{M[1]} & 0 \\ id & \delta[1] & d_N \end{pmatrix}$

$g(\underset{M^{n+1}}{\overset{\cap}{a}}) := (-f(a), a, 0)$

Exercise: g commutes with d.

$r : C(\tau) \to M[1]$

$(a, b, c) \mapsto b$

- $r \circ g = id$ - clear
- ② commutes in $\mathrm{Kom}(R)$ - clear:

$$\overline{\pi} g(a) = \overline{\pi}(-f(a), a, 0) = -f(a)$$

- $g \circ r = id_{C(\tau)}$ in $K(R)$: put $h^n : C(\tau)^n \to C(\tau)^{n-1}$, $(a,b,c) \mapsto (c,0,0)$

then $(g \circ r)(a,b,c) = (-f(b), b, 0)$

$hd(a,b,c) = (a + f(b) + dc, 0, 0)$

$dh(a,b,c) = (-d(c), 0, c)$

$\Rightarrow id - gr = hd + dh.$

- $r \circ \overline{\tau} = \pi$ in $\mathrm{Kom}(R) \Rightarrow \overline{\tau} = g \circ r \circ \overline{\tau} = g \circ \pi$ in $K(R)$, i.e. ① commutes

Lm ~~Let~~ $f\in \mathrm{Hom}_{\mathrm{Kom}(R)}(M^\cdot,N^\cdot)$, $s\in \mathrm{Hom}_{\mathrm{Kom}(R)}(L^\cdot,N^\cdot)$, s-qis. Then $\exists\, g,t$ in $\mathrm{Kom}(R)$

$$\begin{tikzcd}
P^\cdot \arrow[r, dashed, "t"] \arrow[d, dashed, "g"'] & M^\cdot \arrow[d, "f"] \\
L^\cdot \arrow[r, "s"] & N^\cdot
\end{tikzcd}$$

s.t. 1) t-qis

2) diagram commutes in $K(R)$.

Pf.

$$\begin{tikzcd}
C(\tau\circ f)[-1] \arrow[r, "\bar\pi"] \arrow[d, dashed, "g_1^{-1}g_2[-1]"] & M^\cdot \arrow[r, "\tau\circ f"] \arrow[d, "f"'] & C(s) \arrow[r] \arrow[d, "="] & C(\tau\circ f) \arrow[r] \arrow[d, dashed, "g_1^{-1}g_2"] \arrow[dd, dashed, bend left, "\exists g_2"] & M^\cdot[1] \arrow[d, "f{[1]}"] \\
L^\cdot \arrow[r, "s"] & N^\cdot \arrow[r, "\tau"] \arrow[d, "="'] & C(s) \arrow[r, "\pi"] \arrow[d, "="] & L^\cdot[1] \arrow[r, "-s"] \arrow[d, dashed, "\exists g_1"'] & N^\cdot[1] \arrow[d, "="] \\
 & N^\cdot \arrow[r, "\tau"] & C(s) \arrow[r] & C(\tau) \arrow[r] & N^\cdot[1]
\end{tikzcd}$$

iso-sm in $K(R)$ $\Rightarrow \exists g_1^{-1} \rightsquigarrow$ consider $g_1^{-1}g_2[-1]$

s-qis $\overset{\text{l.e.s.}}{\Rightarrow} H^n(C(s))=0\ \forall n\in\mathbb{Z}$.

$\overset{\text{l.e.s.}}{\Rightarrow} H^n(\bar\pi)$ - iso-sm $\Rightarrow \bar\pi$ - qis $\leftarrow$ ⊗Lemma below

Another construction of $D(R)$:

Def. $\underline{D}(R)$ is the category with $\mathrm{Ob}(\underline{D}(R)) := \mathrm{Ob}(\mathrm{Kom}(R))$,

$$\mathrm{Hom}_{D(R)}(M^\cdot,N^\cdot) := \left\{ M^\cdot \xleftarrow{s} P^\cdot \xrightarrow{f} N^\cdot,\ s,f\in K(R),\ s\text{-qis} \right\} \Big/ \ \left(M^\cdot \xleftarrow{s} P^\cdot \xrightarrow{f} N^\cdot\right) \sim \left(M^\cdot \xleftarrow{\tilde s} \tilde P^\cdot \xrightarrow{\tilde f} N^\cdot\right)$$

if there exists $P \xleftarrow{\pi_1} \hat{\tilde P} \xrightarrow{\pi_2} \tilde P$ s.t. $s\circ\pi_1$-qis &

$$\begin{tikzcd}
 & \hat{\tilde P} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
P^\cdot \arrow[d, "s"'] \arrow[drr, "f" near end] & & \tilde P^\cdot \arrow[d, "\tilde f"] \arrow[dll, "\tilde s"' near end] \\
M^\cdot & & N^\cdot
\end{tikzcd}$$

commutes

Ex: equiv. relation (uses lemmas above)

identity: $\mathrm{id}_{M^\cdot} := M^\cdot \xleftarrow{\mathrm{id}} M^\cdot \xrightarrow{\mathrm{id}} M^\cdot$

composition:

$$\begin{tikzcd}
 & & R \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "g"] & & \\
 & P^\cdot \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & Q^\cdot \arrow[dl, "s_2"'] \arrow[dr, "s_2"] & \\
M^\cdot & & N^\cdot & & L^\cdot
\end{tikzcd}$$

$\exists\, t,g$, s.t. t-qis and diagram commutes $\rightsquigarrow$

$$\begin{tikzcd}
 & R^\cdot \arrow[dl, "s_1t"'] \arrow[dr, "f_2g"] & \\
M^\cdot & & L^\cdot
\end{tikzcd}$$

- does not depend on R, t, g:

$$\begin{tikzcd}
 & \tilde R \arrow[dl, dashed, "\tilde t"'] \arrow[dr, dashed, "\tilde g"] & \\
P^\cdot \arrow[dr] & & Q^\cdot \arrow[dl] \\
 & N^\cdot &
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
 & \tilde{\tilde R} \arrow[dl, dashed, "\tilde{\tilde t}"'] \arrow[dr, dashed, "\tilde{\tilde g}"] & \\
R \arrow[dr, "t"'] & & \tilde R \arrow[dl, "\tilde t"] \\
 & P^\cdot &
\end{tikzcd}$$

s.t. $\tilde{\tilde t}$-qis

$$\rightsquigarrow
\begin{tikzcd}
 & \tilde{\tilde R} \arrow[dl, "\tilde{\tilde t}"'] \arrow[dr, "\tilde{\tilde g}"] & \\
R \arrow[d, "s_1t"'] \arrow[drr, "f\circ g" near end] & & \tilde R \arrow[d, "f\circ\tilde g"] \arrow[dll, "s\circ\tilde t"' near end] \\
M & & L^\cdot
\end{tikzcd}$$

- Exercise: does not depend on representatives for $\left\{ M^\cdot \leftarrow P^\cdot \rightarrow N^\cdot \right\}$ & $\left\{ N^\cdot \leftarrow Q^\cdot \rightarrow L^\cdot \right\}$

$$\begin{array}{ccccc}
\mathrm{Kom}(R) & \to & K(R) & \to & \underline{D}(R) \\
M^\cdot & & \longmapsto & & M^\cdot \\
f: M^\cdot\to N^\cdot & & \longmapsto & & M^\cdot \xleftarrow{\mathrm{id}} M^\cdot \xrightarrow{f} N^\cdot
\end{array}$$

Exercise: functor (check composition)

Lm $f, g : M^\circ \to N^\circ$, $s : N^\circ \to L^\circ$, s-qis, $sf = sg$ in $K(R)$ $\Rightarrow$

⊛ $\Rightarrow \exists \underset{\text{- qis}}{t : P^\circ \to M^\circ}$ s.t. $ft = gt$.

Pf. Changing $f \rightsquigarrow f-g$ may assume $g = 0$, and $h^n : M^n \to L^{n-1}$ s.t. $sf = dh + hd$.

$$\begin{tikzcd}
 & & & C(F)[-1] \arrow[dl, "\delta_F[-1]"'] & \\
 & & M^\circ \arrow[dl, "F"'] \arrow[d, "f"] & & \\
 & C(S)[-1] \arrow[r, "\delta_S[-1]"] \arrow[dl, "\delta_F"'] & N^\circ \arrow[r, "s"] & L^\circ \arrow[r, "\delta_s"] & C(s) \\
C(F) & & & &
\end{tikzcd}$$

$F : M^\circ \to C(s)[-1]$

$\underset{\substack{\cap \\ M^n}}{a} \mapsto (\underset{\substack{\cap \\ N^n}}{f(a)}, \underset{\substack{\cap \\ L^{n-1}}}{-h^n(a)})$

Ex: morphism of complexes

$P^\circ := C(F)[-1]$

$t := \delta_F[-1]$

$H^n(C(S)) = 0 \;\forall n \in \mathbb{Z}$ since s-qis (long exact sequence) $\overset{\text{l.e.s.}}{\Rightarrow}$ $\delta_F[-1]$ - qis.

$f \circ \delta_F[-1] = \delta_S[-1] \circ \underbrace{F \circ \delta_F[-1]}_{0} = 0$ □.

Lm $f \sim g : M^\circ \to N^\circ$, $F : Kom(R) \to \mathcal{C}$ inverts qis $\Rightarrow F(f) = F(g)$ in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & X \times [0,1] \arrow[l, bend right, "p"'] \arrow[r, "H"] & Y
\end{tikzcd}$$

$i_0 = i_1 = p^{-1}$ in homot. cat

$\Rightarrow f = Hi_0 = Hi_1 = g$

Pf $Cyl(M^\circ)^n := M^n \oplus M^{n+1} \oplus M^n$

$d : M^n \oplus M^{n+1} \oplus M^n \to M^{n+1} \oplus M^{n+2} \oplus M^{n+1}$

$(a, b, c) \mapsto (da - b, -db, b + dc)$

$i_0, i_1 : M^\circ \to Cyl(M^\circ)$ $\quad$ $p : Cyl(M^\circ) \to M^\circ$

$i_0 : a \mapsto (a, 0, 0)$ $\quad$ $(a, b, c) \mapsto a + c$

$i_1 : a \mapsto (0, 0, a)$

$p \circ i_0 = p \circ i_1 = id_{M^\circ} \Rightarrow H^n(p) \circ H^n(i_0) = H^n(p) \circ H^n(i_1) = id$

$i_1 \circ p \sim id_{Cyl(M)}$ via $h(a, b, c) = (0, a, 0)$

$\Rightarrow H^n(i_1 \circ p) = H^n(i_1) \circ H^n(p) = id$

$\Rightarrow i_0, i_1, p$ - qis. and ~~obtaining~~ $F(i_0) = F(i_1) = F(p)^{-1}$

$f \sim g$ via h $\rightsquigarrow$ $H : Cyl(M^\circ) \to N^\circ$

$(a, b, c) \mapsto f(a) + g(c) - h(b)$ Ex: morphism of complexes

$F(f) = F(H \circ i_0) = F(H) \circ F(i_0) = F(H) \circ F(i_1) = F(H \circ i_1) = F(g)$ □

Thm $\underline{D}(R) \simeq D(R)$

Pf.

$$\begin{tikzcd}
\mathrm{Kom}(R) \arrow[r] \arrow[rr, bend left, "Q"] \arrow[dr, "F"'] & K(R) \arrow[r] \arrow[d, dashed, "G_1"] & \underline{D}(R) \arrow[dl, dashed, "G"] \\
& \mathcal{C} &
\end{tikzcd}$$

G - identity on objects

Every morphism in $\underline{D}(R)$ is $Q(f)\circ Q(s)^{-1}$ for some $f \in \mathrm{Hom}_{\mathrm{Kom}(R)}$, s - qis

$\Rightarrow$ G is unique if exists

$$G\left(M^\circ \xleftarrow{s} P^\circ \xrightarrow{f} N^\circ\right) := F(f)\circ F(s)^{-1}$$

Does not depend on equivalence:

$$\begin{tikzcd}
& & \tilde{\tilde{P}}^\circ \arrow[dl, "u"'] \arrow[dr, "v"] & & \\
& P^\circ \arrow[dl, "s"'] \arrow[drrr, "f"] & & \tilde{P}^\circ \arrow[dlll, "\tilde{s}"'] \arrow[dr, "\tilde{f}"] & \\
M^\circ & & & & N^\circ
\end{tikzcd}$$

$F(\tilde{f})F(\tilde{s})^{-1} = F(\tilde{f})F(v)F(u)^{-1}F(s)^{-1} = F(\tilde{f}v)\circ F(su)^{-1}$

similarly, $F(f)F(s)^{-1} = F(\tilde{f}v)\circ F(su)^{-1}$ □

Exercise: respects composition.

Rk D(R) is additive:

Pf: $M^\circ \xleftarrow{s} P^\circ \xrightarrow{f} N^\circ$, $M^\circ \xleftarrow{\tilde{s}} \tilde{P}^\circ \xrightarrow{\tilde{f}} N^\circ$ ⇝

$$\begin{tikzcd}
& \tilde{\tilde{P}}^\circ \arrow[dl, "\tilde{t}"'] \arrow[dr, "t"] & \\
P^\circ \arrow[dr, "s"'] & & \tilde{P}^\circ \arrow[dl, "\tilde{s}"] \\
& M^\circ &
\end{tikzcd}$$

$\Rightarrow$

$$M^\circ \xleftarrow{s} P^\circ \xrightarrow{f} N^\circ \quad\quad M^\circ \xleftarrow{\tilde{s}} \tilde{P}^\circ \xrightarrow{\tilde{f}} N^\circ$$

$$\wr \quad\quad\quad\quad \wr$$

$$M^\circ \xleftarrow{s\tilde{t}} \tilde{\tilde{P}}^\circ \xrightarrow{f\tilde{t}} N^\circ \quad\quad M^\circ \xleftarrow{\tilde{s}t} \tilde{\tilde{P}}^\circ \xrightarrow{\tilde{f}t} N^\circ$$

$$\rightsquigarrow \quad M^\circ \xleftarrow{s} P^\circ \xrightarrow{f} N^\circ + M^\circ \xleftarrow{\tilde{s}} \tilde{P}^\circ \xrightarrow{\tilde{f}} N^\circ := M^\circ \xleftarrow{s\tilde{t}} \tilde{\tilde{P}}^\circ \xrightarrow{f\tilde{t} + \tilde{f}t} N^\circ$$

Exercise:
- correctly defined.
- additive category.

Def. $\mathrm{Kom}^+(R)$ - full subcat of Kom(R) consisting of M° s.t. $M^n = 0$ $n \ll 0$

$\mathrm{Kom}^-(R)$ — -//- $n \gg 0$

$\mathrm{Kom}^b(R)$ — -//- $|n| \gg 0$

$\rightsquigarrow K^*(R)$, $D^*(R)$, $* = +, -, b, \emptyset$ inverting qis.

Thm 1) R-Mod $\to$ D(R)

$M \mapsto M[0] := (\ldots \to \overset{-1}{0} \to \overset{0}{M} \to \overset{1}{0} \to \ldots)$ - equivalence with the full subcat. of D(R) consisting of M° s.t. $H^n(M^\circ) = 0$, $n \neq 0$

2) $D^*(R) \hookrightarrow D(R)$ - equivalence onto full subcat. of D(R) consisting of M° s.t. $H^n(M) = 0$ $n \ll 0$ $* = +$

$n \gg 0$ $* = -$

$|n| \gg 0$ $* = b$

Rk. $M^\bullet \in D(R)$, the rule $M^\bullet \mapsto \dots \to \overset{-1}{M^{-1}} \to \overset{0}{M^0} \to 0 \to \dots \in D(R)$

is not a functor, e.g.

$$\begin{tikzcd} \cdots \to 0 \to \overset{0}{\mathbb{Z}} \xrightarrow{3} \overset{1}{\mathbb{Z}} \to 0 \to \cdots \arrow[r, mapsto] \arrow[d, phantom, "\wr S"] & \mathbb{Z}[0] \arrow[d, phantom, "\cancel{\wr S}"] \\ 0 \to \mathbb{Z}/2 \to 0 \arrow[r, mapsto] & 0[0] \end{tikzcd}$$

Def $M^\bullet \in D(R)$, $n \in \mathbb{Z}$ ⇝ $\tau^{\ge n} M^\bullet := (\dots \to 0 \to \operatorname{Im} d^{n-1} \to M^n \to M^{n+1} \to \dots)$

$\tau^{\le n} M^\bullet := (\dots \to M^{n-2} \to M^{n-1} \to \ker d^n \to 0 \to \dots)$ – functorial

⇝ $\tau^{\le n} M^\bullet \to M^\bullet$ iso on $H^m(-)$, $m \le n$, $M^\bullet \to \tau^{\ge n} M^\bullet$ iso on $H^m(-)$, $m \ge n$, $f: M^\bullet \to N^\bullet \Rightarrow$

$$\Rightarrow \begin{tikzcd} \tau^{\le n} M^\bullet \arrow[r, "\tau^{\le n} f"] \arrow[d] \arrow[dr, phantom, "G"] & \tau^{\le n} N^\bullet \arrow[d] \\ M^\bullet \arrow[r, "f"'] & N^\bullet \end{tikzcd}, \quad \begin{tikzcd} M^\bullet \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & N^\bullet \arrow[d] \\ \tau^{\ge n} M^\bullet \arrow[r, "\tau^{\ge n} f"'] & \tau^{\ge n} N^\bullet \end{tikzcd}$$

<u>Pf of the Thm:</u>

1) Suppose $H^n(M^\bullet) = 0$, $n \ne 0$ ⇝

$$\begin{tikzcd} \tau^{\le 0} M^\bullet \arrow[r, "\text{qis}"] \arrow[d, "\text{qis}"] & M^\bullet \\ \tau^{\ge 0}\tau^{\le 0} M^\bullet & \end{tikzcd}$$

$\Rightarrow R\text{-Mod} \xrightarrow{\sim} D^0(R)$ – ess. surj.

$D^0(R)$: full subcat of $D(R)$ with $H^n(M) = 0$, $n \ne 0$

$$\begin{tikzcd} & P^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M[0] & & N[0] \end{tikzcd} \sim \begin{tikzcd} & L^\bullet := \tau^{\ge 0}\tau^{\le 0} P^\bullet \arrow[dl, "\tau^{\ge 0}\tau^{\le 0} s"'] \arrow[dr, "\tau^{\ge 0}\tau^{\le 0} f"] & \\ M[0] & & N[0] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & L^{-1} \arrow[r] \arrow[d] & L^0 \arrow[r] \arrow[d] \arrow[dr] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \\ & & & H^0(L^\bullet) \arrow[ul, dashed] \end{tikzcd}$$

, similarly for N. ⇝

$$\begin{tikzcd} & L^\bullet \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & H^0(L^\bullet)[0] \arrow[dl] \arrow[dr] & \\ M[0] & & N[0] \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} & P^\bullet \arrow[dl] \arrow[dr] & \\ M[0] & & N[0] \end{tikzcd} \sim \begin{tikzcd} & H^0(P^\bullet)[0] \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M[0] & & N[0] \end{tikzcd}. \sim \begin{tikzcd} & M[0] \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\beta\alpha^{-1}"] & \\ M[0] & & N[0] \end{tikzcd}$$

⇒ functor $R\text{-Mod} \to D^0(R)$ is surjective on morphisms

injectivity:

$$\begin{tikzcd} & & P^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ & M[0] \arrow[dl, "\mathrm{id}"'] \arrow[drr, "f[0]"] & & M[0] \arrow[d, "g[0]"] \arrow[dlll, "\mathrm{id}"'] \\ M[0] & & & N[0] \end{tikzcd} \overset{\text{as before}}{\sim} \begin{tikzcd} & & H^0(P^\bullet)[0] \arrow[dl, "s"'] \arrow[dr, "t"] & \\ & M[0] \arrow[dl, "\mathrm{id}"'] \arrow[drr, "f[0]"] & & M[0] \arrow[d, "g[0]"] \arrow[dlll, "\mathrm{id}"'] \\ M[0] & & & N[0] \end{tikzcd}$$

$\Rightarrow f = g$.

2) $* = +$. $\underline{D}^+(R) :=$ full subcat. of $D(R)$ with $H^n(M^\bullet) = 0$ $n \ll 0$

$M^\bullet \in \underline{D}^+(R)$, $H^n(M^\bullet) = 0$ $n < n_0$ ⇝ $M^\bullet \to \tau^{\ge n_0} M^\bullet$ – qis $\Rightarrow D^+(R) \to \underline{D}^+(R)$ – ess. surj.

$$\begin{tikzcd} & P^\circ \in Kom(R) \arrow[dl, "qis \to s"'] \arrow[dr, "f"] & \\ M^\circ & & N^\circ \\ & Kom^+(R) \arrow[ul, phantom, "\subset"] \arrow[ur, phantom, "\supset"] & \end{tikzcd}$$

$M^n = N^n = 0$, $n < n_0$

$$\rightsquigarrow \begin{tikzcd} M^\circ \arrow[d, "="] & P^\circ \arrow[l, "qis"'] \arrow[r] \arrow[d, "qis"] & N^\circ \arrow[d, "="] \\ \tau^{\geq n_0} M^\circ & \tau^{\geq n_0} P^\circ \arrow[l, "qis"'] \arrow[r] & \tau^{\geq n_0} N^\circ \end{tikzcd}$$

$\Rightarrow D^+(R) \to \underline{D}^+(R)$ surj on morphisms injective as above.

$* = -, b$ - similar.

Def. R-left-Noetherian ring if $\forall N \leq M \in R\text{-Mod}$ s.t. ~~M~~ M-fin. gen $\Rightarrow N$-fin. generated.

$\rightsquigarrow K_f^*(R)$, $D_f^*(R)$ $* = +, -, b, \emptyset$.

Thm R-left Noetherian ring $\Rightarrow D_f^*(R) \to D^*(R)$ induces an equivalence with the full subcat s.t.

Pf. ~~$\ast$~~ $\underline{D}_f^*(R)$ - respective full subcat. $H^n(M^\circ)$ are fin. gen. and $= 0$ $n \gg 0$ $(* = -)$ $|n| \gg$ $(* = b)$

$* = -$ ess. surjectivity: $M^\circ \in \underline{D}_f^-(R)$, wlog may assume $M^n = 0, n > 0$.

$$\rightsquigarrow \begin{tikzcd} \dots \arrow[r] & \widetilde{M}^{-2} \arrow[r] \arrow[d] & \widetilde{M}^{-1} \arrow[r] \arrow[d] & \widetilde{M}^0 \arrow[r] \arrow[d] & 0 \\ \dots \arrow[r] & M^{-2} \arrow[r] & M^{-1} \arrow[r] & M^0 \arrow[r] & 0 \end{tikzcd}$$

Pick $x_1, x_2, \dots x_{r_0} \in M^0$ s.t. $\{\overline{x_i}\}$ gen. $H^0(M^\circ)$, $\widetilde{M}^0 := \langle x_1, \dots x_{r_0} \rangle \leq M^0$

Pick $x_1, \dots, x_{r_n} \in$ ~~Ker~~ $\ker d^{-n}$ s.t. $\{\overline{x_l}\}$ gen. $H^{-n}(M^\circ)$ & $y_1, \dots y_{l_n} \in M^{-n}$ s.t. $\{d^{-n}(y_l)\}$ gen. $d^{-n}(M^{-n}) \cap \widetilde{M}^{-n+1}$. $\widetilde{M}^{-n} := \langle x_1, \dots x_{r_n}, y_1, \dots, y_{l_n} \rangle \leq M^{-n}$

$\widetilde{M}^\circ \to M^\circ$ - qis, $\widetilde{M}^n$ - fin. gen. $\Rightarrow D_f^-(R) \to \underline{D}_f^-(R)$ - ess. surj.

$$\begin{tikzcd} & P^\circ \arrow[dl] \arrow[dr] & \\ M^\circ & & N^\circ \end{tikzcd} \rightsquigarrow \begin{tikzcd} & \widetilde{P}^\circ \arrow[d] \arrow[ddl, "qis"'] \arrow[ddr, bend left] & \\ & \tau^{\geq n_0} P^\circ \arrow[d] & \\ M^\circ & P^\circ \arrow[l] \arrow[r] & N^\circ \end{tikzcd}$$

$\Rightarrow$ surjective on morphisms injectivity is similar.

$* = b$ - same.

Rk. In general, for $* = +$ or $\emptyset$ the claim is false, but true eg. for R-commut. regular Noeth. of fin Krull dim.

Rk. $\mathcal{C}$-cat, $S \subset Hom_{\mathcal{C}}$ - localizing, $\mathcal{B} \subset \mathcal{C}$ - full subcat. Suppose $S_{\mathcal{B}} := S \cap Hom_{\mathcal{B}}$ is localizing and $\forall s: X' \to X$ in S with $X \in Ob_{\mathcal{B}}$ $\exists f: X'' \to X'$ s.t. $X'' \in Ob_{\mathcal{B}}$ and $s \circ f \in S$. then $\underset{\text{localizations}}{\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{C}[S^{-1}]}$ is fully faithful.

Def $Q \in R\text{-Mod}$ is *injective* if
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, dashed, "\exists h"] \\ & Q & \end{tikzcd}$$
$\forall f$-injective, $\forall g$-hom-sm $\exists h: g = hf$

Lm. R-Mod has *enough injectives*, i.e. $\forall M \in R\text{-Mod}$ $\exists f: M \to Q$ s.t. f-inj. hom-sm & Q-inj. module.

Pf. $\mathbb{Q}/\mathbb{Z}$ - inj. abelian group:
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dashed] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$
use Zorn lemma

$R^\vee := \mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z}) \in R\text{-Mod}$

$\alpha \cdot \varphi := (x \mapsto \varphi(x\alpha))$

$R^\vee$ is injective: $0 \to M \to N \rightsquigarrow \mathrm{Hom}_R(N, R^\vee) \to \mathrm{Hom}(M, R^\vee)$ - surj?
$$\begin{tikzcd} \mathrm{Hom}_R(N, R^\vee) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(M, R^\vee) \arrow[dd, equal] \\ \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \arrow[d, phantom, "\wr"] & \\ \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \end{tikzcd}$$
- surj. since $\mathbb{Q}/\mathbb{Z}$-inj. ab. gr.

$\rightsquigarrow M \xrightarrow{f} \prod_{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})} R^\vee$

$m \mapsto \prod_{\varphi \in \mathrm{Hom}} (\underset{R}{\underset{\cap}{x}} \mapsto \underset{\mathbb{Q}/\mathbb{Z}}{\underset{\cap}{\varphi(xm)}})$

<u>Exercise</u>:
- f-injective hom-sm of R-Mod.
- $\prod_{\varphi \in \mathrm{Hom}} R^\vee$ is injective

Lm $M^\circ \in \mathrm{Kom}^+(R) \Rightarrow \exists Q^\circ$ s.t. Q^n-inj. $\forall n \in \mathbb{Z}$ & $s: M^\circ \to Q^\circ$ - qis.

Pf. May assume $M^n = 0$, $n < 0$. Construct Q^n & s^n inductively.

$n = 0, 1$:
$$\begin{tikzcd} 0 \arrow[r] & M^0 \arrow[r] \arrow[d, hook, "s^0"] & M^1 \arrow[r] \arrow[dr, "s^1"] \arrow[dd] & \ldots & \\ 0 \arrow[r] & Q^0 \arrow[rr] \arrow[dr] & & Q^1 & \\ & & Q^0 \oplus_{M^0} M^1 \arrow[ur, hook] & & \end{tikzcd}$$
enough inj. → s^0

$Q^0 \oplus M^1 / (s^0, d^0)(M^0) =: Q^0 \oplus_{M^0} M^1$

enough inj.

$$\begin{tikzcd} & M^n \arrow[r] \arrow[d, "s^n"'] \arrow[dr] & M^{n+1} \arrow[rrd, "s^{n+1}"] \arrow[dr] & & \\ Q^{n-1} \arrow[r, "d"] & Q^n \arrow[r, two heads] & Q^n/dQ^{n-1} \arrow[rr] \arrow[r] & Q^n/dQ^{n-1} \oplus_{M^n} M^{n+1} \arrow[r, hook] & Q^{n+1} \end{tikzcd}$$
enough inj.

<u>Exercise</u>: s - qis.

Lm. $s: Q^\circ \to M^\circ$ - qis, $Q^\circ \in K^+(R\text{-Inj})$, $M^\circ \in K^+(R) \Rightarrow \exists t: M^\circ \to Q^\circ$ s.t. $t \circ s = \mathrm{id}_{Q^\circ}$ in $K(R)$

Pf $Q^\circ \xrightarrow{s} M^\circ \xrightarrow{\tau} C(s) \xrightarrow{\pi} Q^\circ[1]$. s-qis $\Rightarrow H^n(C(s)) = 0$ $\forall n \in \mathbb{Z}$.

<u>Claim</u>: $\pi \sim 0$: may assume $Q^n = 0 = C^n := C(s)^n$ $\forall n \le 0$.

construct homotopy inductively.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d, "0"'] & C^0 \arrow[r] \arrow[dl, "0"'] \arrow[d, "\pi^0"] & C^1 \arrow[r] \arrow[dl, dashed, "\exists h^1"] & C^2 \arrow[r] & \cdots \\
0 \arrow[r] & Q^0 \arrow[r] & Q^1 \arrow[r] & Q^2 \arrow[r] & Q^3 \arrow[r] & \cdots
\end{tikzcd}$$

$\exists h^1$ s.t. $\pi^0 = h^1 d^0$ since d^0 is inj. & $Q^1 \in R\text{-Inj}$.

$$\begin{tikzcd}
C^{n-2} \arrow[r] \arrow[d] & C^{n-1} \arrow[r, "d^{n-1}"] \arrow[dl, "h^{n-1}"'] \arrow[d, "\pi^{n-1}"] & C^n \arrow[d] \arrow[dl, dashed, "?"'] \\
Q^{n-1} \arrow[r, "d"] & Q^n \arrow[r] & Q^{n+1}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
C^{n-2} \arrow[r] & C^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr, two heads] \arrow[dd, "\pi^{n-1}-dh^{n-1}"'] & & C^n \arrow[ddll, dashed, bend left, "h^n"] \\
 & & \operatorname{Im} d^{n-1} \arrow[ur, hook] \arrow[dl, dashed, "f"] & \\
 & Q^n & &
\end{tikzcd}$$

univ. property of Q^n

$$(\pi^{n-1}-dh^{n-1})(d\overset{C^{n-2}}{\overset{\in}{c}}) =$$

$$= \pi dc - dhdc =$$

$$= d\pi c - dhdc = d(\pi - hd)c \overset{\text{previous step}}{=}$$

$$= ddhc = 0 \xRightarrow[\ker d^{n-1} = dC^{n-2}]{} \exists f.$$

$\Rightarrow h^n d^{n-1} = \pi^{n-1} - dh^{n-1}$, i.e. $\pi^{n-1} = h^n d^{n-1} + dh^{n-1}$

$C(S)^n = Q^{n+1} \oplus M^n$, $\pi^n: Q^{n+1} \oplus M^n \xrightarrow{(id,0)} Q^{n+1}$, $h^n: Q^{n+1} \oplus M^n \xrightarrow{(r,t)} Q^n$, $d^n: Q^{n+1} \oplus M^n \xrightarrow{\begin{pmatrix}-d & 0\\ s & d\end{pmatrix}} Q^{n+2} \oplus M^{n+1}$, $d^n_{Q}: Q^{n+1} \xrightarrow{-d} Q^{n+2}$

$\pi = hd + dh \Rightarrow$

$\Rightarrow (id, 0) = (-rd + ts, td) + (dr, dt)$

$\Rightarrow id = -rd + dr + ts \quad \Rightarrow \quad id \sim ts.$

$ 0 = td - dt \qquad\qquad t$ - hom-sm of complexes

<u>Cor.</u> $Q^\bullet \in K^+(R\text{-Inj})$, $M^\bullet \in K^+(R) \Rightarrow \operatorname{Hom}_{K^+(R)}(M^\bullet, Q^\bullet) \simeq \operatorname{Hom}_{D(R)}(M^\bullet, Q^\bullet)$.

PS. One may construct $D(R)$ using $\operatorname{Hom}_{D(R)}(M^\bullet, N^\bullet) := \left\{ \begin{tikzcd} M^\bullet \arrow[dr, "f"'] & & N^\bullet \arrow[dl, "s"', "qis"] \\ & L^\bullet & \end{tikzcd} \right\} / \sim$ $\quad s^{-1}f$

(Exercise).

$$\begin{tikzcd}
M^\bullet \arrow[dr, "f"'] & & Q^\bullet \arrow[dl, "s"] \\
 & L^\bullet &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
M^\bullet \arrow[dr, "f"'] \arrow[drr, "tf"] & & Q^\bullet \arrow[dl, "s"'] \arrow[d, "id"] \\
 & L^\bullet \arrow[dr, "t"'] & Q^\bullet \arrow[d, "id"] \\
 & & Q^\bullet
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
M^\bullet \arrow[dr, "f"'] & & Q^\bullet \arrow[dl, "s"] \\
 & L^\bullet &
\end{tikzcd}
\sim
\begin{tikzcd}
M^\bullet \arrow[dr, "tf"'] & & Q^\bullet \arrow[dl, "id"] \\
 & Q^\bullet &
\end{tikzcd}$$

from Lemma $\rightarrow t$

$\Rightarrow \operatorname{Hom}_{K^+(R)} \to \operatorname{Hom}_{D^+(R)}$ is surjective.

Injectivity:

$$\begin{tikzcd}
M^\bullet \arrow[dr, "f"'] & & Q^\bullet \arrow[dl, "id"] \\
 & Q &
\end{tikzcd}
\sim
\begin{tikzcd}
M^\bullet \arrow[dr, "g"'] & & Q^\bullet \arrow[dl, "id"] \\
 & Q^\bullet &
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
M^\bullet \arrow[d, "f"'] \arrow[drr, "g"] & & Q^\bullet \arrow[dll, "id"'] \arrow[d, "id"] \\
Q^\bullet \arrow[dr, "\tilde{s}"'] & & Q^\bullet \arrow[dl, "s\text{-qis}"] \\
 & L^\bullet &
\end{tikzcd}$$

$\Rightarrow s = \tilde{s} \Rightarrow sg = sf$; s-qis $\Rightarrow \exists t: L^\bullet \to Q^\bullet$ s.t. $t \circ s = id$ in $K^+(R)$

$\Rightarrow g = tsg = tsf = f$ □

<u>Thm.</u> $K^+(R\text{-Inj}) \to D^+(R)$ - equiv. of cats.

<u>Pf</u>: essentially surjective since $M^\bullet \xrightarrow{qis} Q^\bullet$

fully faithful by the Corollary, $\operatorname{Hom}_{K^+(R\text{-Inj})}(Q^\bullet, \tilde{Q}^\bullet) \simeq$

$\operatorname{Hom}_{D^+(R)}(Q^\bullet, \tilde{Q}^\bullet)$ □

Def $P \in R\text{-Mod}$ is *projective* if $\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] & \\ M \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$ $\forall f$-surj $\forall g$ $\exists h$ s.t. $g = fh$.

Thm. $K^-(R\text{-Proj}) \to D^-(R)$ - equiv. of cats

Pf. as above.

Def. $F: R\text{-Mod} \to \mathcal{Ab}$ (or S-Mod, or any abelian cat.) is

- left exact if $0 \to M \to N \to L$ - exact $\Rightarrow 0 \to F(M) \to F(N) \to F(L)$ is exact
- right exact if $M \to N \to L \to 0$ - exact $\Rightarrow F(M) \to F(N) \to F(L) \to 0$ is exact.

Ex: $\mathrm{Hom}(M, -): R\text{-Mod} \to \mathcal{Ab}$ is left exact

$M \otimes_R - : R\text{-Mod} \to \mathcal{Ab}$ is right exact

right R-Mod.

Def. $F: R\text{-Mod} \to \mathcal{Ab}$ - additive $\rightsquigarrow K^*(F): K^*(R) \to K^*(\mathcal{Ab})$, $* = +, -, b, \emptyset$

$$\rightsquigarrow \begin{tikzcd} K^*(R) \arrow[r, "K^*(F)"] \arrow[d] & K^*(\mathcal{Ab}) \arrow[d] \\ D^*(R) \arrow[r, dashed] & D^*(\mathcal{Ab}) \end{tikzcd}$$

exists only if $K^*(F)$ maps qis to qis (or, equivalently, acyclic complexes to acyclic)

Ex: $\mathrm{Hom}(\mathbb{Z}/2, -)$: $\dots 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \dots \rightsquigarrow 0$

$\wr$

$\dots 0 \to 0 \to \mathbb{Z}/2 \to 0 \dots \rightsquigarrow \dots 0 \to 0 \to \mathbb{Z}/2 \to 0 \dots$

$$\rightsquigarrow \begin{tikzcd} K^+(R) \arrow[r, "K^+(F)"] \arrow[d, "Q_R"] & K^+(\mathcal{Ab}) \arrow[d, "Q_{\mathcal{Ab}}"] \\ D^+(R) \arrow[r, dashed, "RF"] \arrow[d, dashed, bend left, "\Phi"] & D^+(\mathcal{Ab}) \\ K^+(R\text{-Inj}) \arrow[u, "i", "\simeq"'] \arrow[uu, bend left=60, "j"] & \end{tikzcd}$$

Φ - quasi-inv to i

$\rightsquigarrow RF := Q_{\mathcal{Ab}} \circ K^+(F) \circ j \circ \Phi : D^+(R) \to D^+(\mathcal{Ab})$ - *right derived* functor of F

Rk There is a canonical (natural transformation) morphism $Q_{\mathcal{Ab}} \circ K^+(F) \to RF \circ Q_R$, which is a universal one, i.e. RF is the left Kan extension of $Q_{\mathcal{Ab}} \circ K^+(F)$ along Q_R.

Rk. One can $+ \rightsquigarrow \emptyset$ using "K-injective" complexes

Def $R^n F(M^\cdot) := H^n(RF(M^\cdot)) \in \mathcal{Ab}$ - *n-th right derived functor*

$M^\cdot \xrightarrow{qis} Q^\cdot \rightsquigarrow R^n F = H^n(\dots \to F(Q^{n-1}) \to F(Q^n) \to \dots)$.

does not depend on the choice of $Q^\cdot$: $\begin{tikzcd} M \arrow[r, "qis"] \arrow[dr, "qis"'] & Q^\cdot \arrow[d, dashed, "qis"] \\ & \tilde{Q}^\cdot \end{tikzcd}$ $\leftarrow$ by corollary, $\mathrm{Hom}_{D(R)}(M^\cdot, \tilde{Q}^\cdot) = \mathrm{Hom}_{D(R)}(Q^\cdot, \tilde{Q}^\cdot)$

iso in $K^+(R)$

$\Rightarrow K^+(F)(Q^\cdot) \xrightarrow{\sim} K^+(F)(\tilde{Q}^\cdot)$

$\mathrm{Hom}_{K^+(R)}(M^\cdot, \tilde{Q}^\cdot) \xrightarrow{\sim} \mathrm{Hom}_{K^+(R)}(Q^\cdot, \tilde{Q}^\cdot)$

$M \in R\text{-Mod} \Rightarrow R^n F(M) := R^n F(M[0])$

Lm. F-left exact. $\Rightarrow$ 1) $R^0F(M) \simeq F(M)$ 2) $R^nF(M)=0$, $n<0$

3) $0\to M\to N\to L\to 0$ - s.e.s. $\Rightarrow \exists$ long exact sequence

$$0\to R^0F(M)\to R^0F(N)\to R^0F(L)\to R^1F(M)\to \dots$$

Pf. $M \rightsquigarrow M[0] \xrightarrow{qis} Q^\bullet$,

$$\begin{tikzcd} & -1 & 0 & 1 & \\ & 0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \arrow[d] & \\ & 0 \arrow[r] & Q^0 \arrow[r, "d^0"] & Q^1 \arrow[r] & \dots \end{tikzcd}$$

$\rightsquigarrow 0\to F(Q^0)\to F(Q^1)\to \dots$

$M=\ker d^0 \overset{\text{left exact}}{\Longrightarrow} F(M)=\ker F(d^0) = R^0F(M) \Rightarrow 1)$; 2) is clear.

3)

$$\begin{tikzcd} & & L[0] \\ M[0] \arrow[r, "i"] \arrow[d] & N[0] \arrow[r] \arrow[d] & C(i) \arrow[u, dashed] \arrow[d, dashed, "\leftarrow qis\ (5\text{-lemma} + l.e.s)"] \\ Q^\bullet_M \arrow[r, "Q_i"] & Q^\bullet_N \arrow[r] & C(Q_i) \in K^+(R\text{-}Inj) \end{tikzcd}$$

; corollary $\Rightarrow$

$$\begin{tikzcd} C(i) \arrow[r] \arrow[rd, "qis"'] & L[0] \arrow[d, dashed, "qis"] \\ & C(Q_i) \end{tikzcd}$$

i.e. $C(Q_i)$ is an inj. resolution of $L[0]$.

$$\begin{tikzcd} \rightsquigarrow F(Q^\bullet_M) \arrow[r, "F(Q_i)"] & F(Q^\bullet_N) \arrow[r] \arrow[rd] & F(C(Q_i)) \arrow[d, phantom, "\wr"] \\ & & C(F(Q_i)) \end{tikzcd}$$

$\wr \leftarrow$ by construction of Cone.

$\Rightarrow$ l.e.s.

$$\begin{tikzcd} H^n(F(Q^\bullet_M)) \arrow[r] \arrow[d, equal] & H^n(F(Q^\bullet_N)) \arrow[r] \arrow[d, equal] & H^n(F(C(Q_i))) \arrow[r] \arrow[d, phantom, "\wr"] & H^{n+1}(F(Q^\bullet_M)) \arrow[d, equal] \\ R^nF(M) \arrow[r] & R^nF(N) \arrow[r] & R^nF(L) \arrow[r] & R^{n+1}F(M) \end{tikzcd}$$ □

Rk. Same for left derived functors,

$$\begin{tikzcd} K^-(R) \arrow[r] \arrow[d] & K^-(Ab) \arrow[d] \\ D^-(R) \arrow[r, dashed] \arrow[u, bend left] & D^-(Ab) \\ K^-(R\text{-}Proj) \arrow[u, "\sim"] \arrow[u, dashed, leftarrow] & \end{tikzcd}$$

right Kan extension; universal transformation

$$LF\circ Q_R \to Q_{Ab}\circ K^-(F)$$

Rk. Instead of inj. modules one can use a subcat I_F of adapted to F modules, i.e.

1) I_F is closed under finite $\oplus$

2) If $M^\bullet\in K^+(R)$ is acyclic with $M^n\in I_F$ $\forall n$ then $F(M^\bullet)$ is acyclic

3) $\forall M\in R\text{-Mod}$ $\exists$ inj map $f: M\hookrightarrow Q$ for some $Q\in I_F$

$$\begin{tikzcd} \rightsquigarrow K^+(R) \arrow[r] \arrow[d] & K^+(Ab) \arrow[d] \\ D^+(R) \arrow[r, dashed] & D^+(Ab) \\ K^+(I_F) \arrow[u, "\simeq"] \arrow[u, dashed, leftarrow] \arrow[uu, bend left] & \end{tikzcd}$$

Rk. R-Inj are F-adapted $\forall$ F-left-exact.

Def. $Ext^n(M,N) := Hom_{D(R)}(M[0], N[n])$.

Thm. 1) $Ext^n(M,N) \simeq (R^nHom(M,-))(N)$, in particular, $Ext^n_R(M,N) = \begin{cases} 0, & n<0 \\ Hom(M,N), & n=0 \end{cases}$

2) $Ext^1(M,N) \simeq \{\underbrace{0\to N\to L\to M\to 0}_{\text{- s.e.s.}}\}/$ s.e.s are equiv. if $\exists\theta$

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "="] & L' \arrow[r] \arrow[d, "\theta", "\simeq"'] & M \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & N \arrow[r] & L \arrow[r] & M & \end{tikzcd}$$

Pf. 1) $Hom_{D(R)}(M, N[n]) \underset{N\xrightarrow{qis}Q^\bullet}{\simeq} Hom_{D(R)}(M, Q^\bullet[n]) \simeq Hom_{K(R)}(M, Q^\bullet[n])$

$$\begin{tikzcd} & 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] \arrow[ld, dashed] & 0 \arrow[d] \\ & Q^{n-1} \arrow[r] & Q^n \arrow[r] & Q^{n+1} \end{tikzcd}$$

$\{f\in Hom(M,Q^n) \mid df=0\}/dHom(M,Q^{n-1}) = H^n(RHom(M,-)(Q^\bullet))$

2)

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & L \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \rightsquigarrow \begin{tikzcd} & -1 & 0 \\ & N \arrow[r] & L \arrow[ld] \arrow[rd] & \\ 0 \arrow[r] & M & & N \arrow[r] & 0 \end{tikzcd}$$

;

$$\begin{tikzcd} & P^\bullet \arrow[ld, "qis"'] \arrow[rd] & \\ M[0] & & N[1] \end{tikzcd} \sim \begin{tikzcd} & \tau^{\geq 0}\tau^{\leq 0}P \arrow[ld] \arrow[rd, "f"] & \\ N[0] & & N[1] \end{tikzcd}$$

, $\tau^{\geq 0}\tau^{\leq 0}P = \dots\to 0\to Im\,d_P^{-1} \overset{i}{\hookrightarrow} P^0\to 0\dots$

$$\sim \begin{tikzcd} 0 \arrow[r] & N \arrow[r] & (N\oplus P^0)/(f,i) \arrow[r] \arrow[ld] \arrow[rd] & 0 \\ & M[0] & & N[1] \end{tikzcd}$$

$\Rightarrow$ surjective

Exercise: injective

Example: $R := \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix} \subseteq M_3(\mathbb{C}) \simeq \mathbb{C}[\bullet \leftarrow \bullet \leftarrow \bullet]$

Claim. $R\text{-Mod} \simeq \mathrm{Rep}_{\mathbb{C}}(\bullet \leftarrow \bullet \leftarrow \bullet)$, $\mathrm{Ob} = \left\{ V_1 \xleftarrow{\varphi_{12}} V_2 \xleftarrow{\varphi_{23}} V_3 \;\middle|\; \begin{array}{l} V_i \in \mathbb{C}\text{-Vect} \\ \varphi_{12}: V_2 \to V_1 \\ \varphi_{23}: V_3 \to V_2 \end{array} \right\}$

$\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{C}}} =$
$$\begin{tikzcd} V_1 \arrow[d, "\delta_1"] & V_2 \arrow[l, "\varphi_{12}"'] \arrow[d, "\delta_2"] & V_3 \arrow[l, "\varphi_{23}"'] \arrow[d, "\delta_3"] \\ W_1 & W_2 \arrow[l, "\psi_{12}"'] & W_3 \arrow[l, "\psi_{23}"'] \end{tikzcd}$$

$M \in R\text{-Mod} \longmapsto (e_{11}M \xleftarrow{e_{12}} e_{22}M \xleftarrow{e_{23}} e_{33}M)$, e_{ij} - matrix units

$V_1 \oplus V_2 \oplus V_3 \longleftarrow (V_1 \xleftarrow{\varphi_{12}} V_2 \xleftarrow{\varphi_{23}} V_3)$

e_{ii} act as projectors on V_i,
$e_{12} \leftrightarrow \varphi_{12}$, $e_{23} \leftrightarrow \varphi_{23}$, $e_{13} = e_{12}e_{23}$

Indecomposable modules:

$S_1 = P_1 := \mathbb{C} \leftarrow 0 \leftarrow 0$ $\qquad 0 \leftarrow \mathbb{C} \xleftarrow{id} \mathbb{C} =: I_2$

$P_2 := \mathbb{C} \xleftarrow{id} \mathbb{C} \leftarrow 0$ $\qquad 0 \leftarrow \mathbb{C} \leftarrow 0 =: S_2$

$I_1 = P_3 := \mathbb{C} \xleftarrow{id} \mathbb{C} \xleftarrow{id} \mathbb{C}$ $\qquad 0 \leftarrow 0 \leftarrow \mathbb{C} =: S_3 = I_3$

P_1, P_2, P_3 - projective: $P_1 \oplus P_2 \oplus P_3 \simeq R$

I_1, I_2, I_3 - injective: e.g. for I_2:

$$\begin{tikzcd} V_1 \arrow[d, hook] \arrow[rrrd, "\delta_1"] & V_2 \arrow[l] \arrow[d, hook] \arrow[rrrrd, "\delta_2"] & V_3 \arrow[l] \arrow[d, hook] \arrow[rrrrd, "\delta_3"] & & & \\ W_1 \arrow[rrr, bend right, "0"'] & W_2 \arrow[l] \arrow[rrrr, bend right, "g_2 = \text{extend } \delta_2 \text{ arbitrary}"'] & W_3 \arrow[l, "\psi_{23}"'] \arrow[rrr, bend right, "g_2 \circ \psi_{23}"'] & 0 & \mathbb{C} \arrow[l] & \mathbb{C} \arrow[l, "id"'] \end{tikzcd}$$

S_1, S_2, S_3 - simple.

$T := P_2 \oplus P_3 \oplus S_2 \qquad \mathrm{End}_R(T) = \begin{pmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & 0 & \mathbb{C} \end{pmatrix}$

Claim $\mathrm{Ext}^n(T,T) = 0$, $n \neq 0$

$\mathrm{Ext}^n(T, P_3) = 0$, $n \neq 0$ since P_3 is injective

$\left.\begin{array}{l} P_2[0] \simeq (I_1 \to I_3) \\ S_2[0] \simeq (I_2 \to I_3) \end{array}\right\} \rightsquigarrow \begin{array}{l} 0 \to \mathrm{Hom}_R(T, I_1) \twoheadrightarrow \mathrm{Hom}_R(T, I_3) \to 0 \\ 0 \to \mathrm{Hom}_R(T, I_2) \twoheadrightarrow \mathrm{Hom}_R(T, I_3) \to 0 \end{array}$

$\mathrm{Hom}_R(T, I_3) \simeq \mathbb{C} \leftarrow$ projection $T \to P^3 \xrightarrow{\alpha} I_3$

Rk. P_2, P_3, S_2 - exceptional ~~objects~~ modules, i.e. $\mathrm{Ext}^n(M,M) = \begin{cases} 0, & n \neq 0 \\ \mathbb{C}, & n = 0 \end{cases}$

$\{P_2, P_3, S_2\}$ - strong exceptional collection, i.e. $\mathrm{Ext}^n(M_i, M_j) = 0$, $n \neq 0$ & M_i - exceptional

$\tilde{R} := \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{pmatrix} \rightsquigarrow \mathrm{End}_R(T) \simeq \tilde{R}^{op}$.

Claim $D^b_f(\tilde{R}) \simeq D^b_f(R) \qquad T \overset{L}{\underset{\tilde{R}}{\otimes}} - \dashv \mathrm{RHom}_R(T, -)$.

$M^\bullet \longmapsto {}_R T \overset{L}{\underset{\tilde{R}}{\otimes}} M^\bullet$ $\qquad$ unit of the adjunction: $M^\bullet \longrightarrow \mathrm{RHom}_R(T, T \overset{L}{\otimes} M^\bullet) \simeq \mathrm{RHom}_R(T,T) \otimes M^\bullet \simeq \tilde{R} \otimes M^\bullet \simeq M^\bullet$

~~every~~ enough projectives, $M^\bullet$ - fin. gen.

$\mathrm{RHom}_R(T, N^\bullet) \longleftarrow N^\bullet$ $\qquad \Rightarrow T \overset{L}{\underset{R}{\otimes}} -$ is fully faithful.

$P_2 \oplus P_3 \oplus S_2 \Rightarrow P_2, P_3, S_2$ are in the image; $R \hookrightarrow P_3 \oplus P_2 \oplus P_2 \twoheadrightarrow S_2$ - s.e.s

$\Rightarrow R$ in the image $\xrightarrow[\text{every module has finite proj. res.}]{}$ essentially surjective

Rk. R-Mod $\not\simeq$ $\tilde{R}$-Mod. One has $\tilde{R}$-Mod $\simeq$ $Rep_{\mathbb{C}}(\bullet \to \bullet \leftarrow \bullet)$, and here there are no indecomposable s.t. it is both projective & injective, while $P_3 \in R\text{-Inj} \wedge R\text{-Proj}$.

Def $\mathcal{C}$-category. $\mathcal{C}$-additive if $\forall A, B \in Ob\,\mathcal{C}$ $Hom_{\mathcal{C}}(A,B)$ is an abelian group and

1) $Hom_{\mathcal{C}}(B,C) \times Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{C}}(A,C)$ is bilinear

2) $\exists$ an object $0 \in Ob\,\mathcal{C}$ s.t. $Hom_{\mathcal{C}}(0,A) = Hom_{\mathcal{C}}(A,0) = \{0\}$ $\forall A \in Ob\,\mathcal{C}$

3) $\forall A, B \in Ob\,\mathcal{C}$ $\exists$ $A \amalg B$ & $A \times B$ and the canonical morphism $A \amalg B \to A \times B$ is an isomorphism

$$\begin{tikzcd} A \arrow[r] \arrow[d, "id_A"'] & A \amalg B \arrow[dl, dashed] & B \arrow[l] \arrow[dll, "0"] \\ A & & \end{tikzcd} , \quad \begin{tikzcd} A \arrow[r] \arrow[dr, "0"'] & A \amalg B \arrow[d, dashed] & B \arrow[l] \arrow[dl, "id"] \\ & B & \end{tikzcd} \rightsquigarrow \begin{tikzcd} & A \amalg B \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ A & A \times B \arrow[l] \arrow[r] & B \end{tikzcd}$$

$\mathcal{C}, \mathcal{C}'$-additive cats. $F: \mathcal{C} \to \mathcal{C}'$ is additive if $Hom_{\mathcal{C}}(A,B) \xrightarrow{F} Hom_{\mathcal{C}'}(F(A), F(B))$ is a homom $\forall A, B$.

Rk. Additiveness of a cat is a property, not structure: 3) $\Rightarrow$ $\exists$ structure of ab. monoid on hom-sets

Def. k-field, $\mathcal{C}$-additive. $\mathcal{C}$ is k-linear if $\forall A, B \in Ob\,\mathcal{C}$ $Hom_{\mathcal{C}}(A,B)$ is equipped with a structure of a k-vector space and $\forall A,B,C$ $Hom_{\mathcal{C}}(B,C) \times Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{C}}(A,C)$

$\mathcal{C}, \mathcal{C}'$-k-linear, $F: \mathcal{C} \to \mathcal{C}'$ is k-linear if $Hom_{\mathcal{C}}(A,B) \xrightarrow{F} Hom_{\mathcal{C}'}(F(A), F(B))$ is bilinear

is k-linear $\forall A, B \in Ob\,\mathcal{C}$.

Def. $\mathcal{C}$-additive $\rightsquigarrow$ homotopy category

$Kom^*(\mathcal{C}) \rightsquigarrow K^*(\mathcal{C})$, $* = b, +, -, \emptyset$

Def $\mathcal{C}$-additive. $\mathcal{C}$-abelian if

1) $\forall A, B$, $f: A \to B$ $\exists$ ker f & coker f

$$\begin{tikzcd} \ker f \arrow[r, "i"] & A \arrow[r, "f"] & B \\ C \arrow[u, dashed, "\exists!"] \arrow[ur, "i'"] \arrow[urr, bend right, "0"'] & & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & B \arrow[r, "\pi"] \arrow[dr, "\pi'"] & coker \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

2) $\forall f: A \to B$

$$\begin{tikzcd} \ker f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "\pi"] & coker f \\ & coker\, i \arrow[ur, dashed] \arrow[r, "\simeq"] \arrow[urr, bend right, "0"'] & \ker \pi \arrow[u] & \end{tikzcd}$$

Def. $\mathcal{C}$-abelian $\rightsquigarrow D^*(\mathcal{C})$, $* = b, +, -, \emptyset$.

$$D^*(\mathcal{C}) = Kom^*(\mathcal{C})[qis^{-1}] \simeq K^*(\mathcal{C})[qis^{-1}]$$

via sequences of roofs — via roofs.

If $\mathcal{C}$ has enough injectives $\Rightarrow D^+(\mathcal{C}) \simeq K^+(\mathcal{C}\text{-Inj})$

projectives $\Rightarrow D^-(\mathcal{C}) \simeq K^-(\mathcal{C}\text{-Proj})$

If $\mathcal{C}$ - k-linear abelian $\Rightarrow K^*(\mathcal{C}), D^*(\mathcal{C})$ are k-linear.

Def $\mathcal{D}$-additive. $\mathcal{D}$ is **triangulated** if it is equipped with an additive equivalence $[1]: \mathcal{D} \to \mathcal{D}$ (**shift** functor) & a set of **distinguished triangles**, i.e. sequences $A \to B \to C \to A[1]$ satisfying

TR1
- $A \xrightarrow{id} A \to 0 \to A[1]$ is dist. tr.
-

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\wr"] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[r] \arrow[d, "\wr"] & A[1] \arrow[d, "\wr"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

$A \to B \to C \to A[1]$ – dist. tr. $\Rightarrow$ $A' \to B' \to C' \to A'[1]$ – dist. tr.

- $\forall f: A \to B$ $\exists$ dist. tr. $A \xrightarrow{f} B \to C \to A[1]$

TR2 $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ – dist. tr. $\Longleftrightarrow$ $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ is dist. tr.

TR3

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "\exists h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

& ... dist.

TR4 "Octahedral axiom"

dist.? tr.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "="] & B \arrow[r, "\tau_f"] \arrow[d, "g"] & C_f \arrow[r, "\pi_f"] \arrow[d, dashed, "u"] & A[1] \arrow[d, "="] \\ A \arrow[r, "gf"] & C \arrow[r, "\tau_{gf}"] \arrow[d, "\tau_g"] & C_{gf} \arrow[r, "\pi_{gf}"] \arrow[d, dashed, "v"] & A[1] \\ & C_g \arrow[r, "id"] \arrow[d, "\pi_g"] & C_g \arrow[d, dashed, "w"] & \\ & B[1] \arrow[r, "\tau_f[1]"] & C_f[1] & \end{tikzcd}$$

$\rightsquigarrow$ $\exists u, v, w$ s.t. everything commutes & the second column is a dist. tr.

Rk TR2&TR3 $\rightsquigarrow$

$$\begin{tikzcd} A \arrow[r] \arrow[d, dashed, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, dashed, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

&

Rk TR1&TR3 $\rightsquigarrow$ $A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$ is dist $\Rightarrow$ $g \circ f = 0$.

Def $\mathcal{C}, \mathcal{C}'$ – tr. cats. An additive $F: \mathcal{C} \to \mathcal{C}'$ is **exact** if

1) there is an is-sm of functors $F \circ [1]_{\mathcal{C}} \xrightarrow{\sim} [1]_{\mathcal{C}'} \circ F$

2) $A \to B \to C \to A[1]$ – dist. $\Rightarrow$

$$\begin{tikzcd} F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r, dashed] \arrow[dr, bend right] & F(A)[1]_{\mathcal{C}'} \\ & & & F(A[1]_{\mathcal{C}}) \arrow[u, "\wr"] \end{tikzcd}$$

is dist.

Thm $\mathcal{C}$-additive $\Rightarrow$ $K^*(\mathcal{C})$
$\mathcal{C}$-abelian $\Rightarrow$ $D^*(\mathcal{C})$, $* = b, +, -, \phi$, is triangulated with $[1]$ being shift of complexes & dist. triangles being the ones isomorphic to $M^\bullet \xrightarrow{f} N^\bullet \xrightarrow{\tau_f} C(f) \xrightarrow{\pi_f} M^\bullet[1]$ (in $K^*(\mathcal{C})$ or $D^*(\mathcal{C})$ respectively)

$\mathcal{C}$-abelian $\Rightarrow$ $K^*(\mathcal{C}) \to D^*(\mathcal{C})$ is an exact functor.

Pf exercise

Rk SH – triangulated category

Lm $\mathcal{D}$ – triangulated, $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is a dist. triangle $\Rightarrow$ $\forall M \in \mathrm{Ob}\,\mathcal{D}$

$\mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ & $\mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)$ are exact sequences.

Pf. $\forall \varphi: M \to B$ s.t. $g \circ \varphi = 0$ ⇝
$$\begin{tikzcd} M \arrow[r, "id"] \arrow[d, dashed, "\exists \psi"'] & M \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d, "0"] & M[1] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & A[1] \end{tikzcd}$$
$\to f \circ \psi = \varphi$.

The other sequence is similar □.

Rk. R-ring ⇝ $\mathrm{Hom}_{D(R)}(R, M^\circ[n]) \simeq \mathrm{Hom}_{K(R)}(R, M^\circ[n]) \simeq H^n(M^\circ)$

⇝ l.e.s. of cohomology for $A \xrightarrow{f} B \xrightarrow{g} C(f) \xrightarrow{\pi_f} A[1]$ is the above lemma applied to $M := R[0]$.

Def. $\mathcal{T}$ - triang. cat, $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ - strictly full triang. subcats.

- $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ - (orthogonal) decomposition if
  1) $\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 = \mathrm{Hom}_{\mathcal{T}}(A_2, A_1)$ $\forall A_1 \in \mathcal{T}_1, A_2 \in \mathcal{T}_2$.
  2) $\forall A \in \mathcal{T}$ $\exists A_1 \in \mathcal{T}_1, A_2 \in \mathcal{T}_2$ s.t. $A \simeq A_1 \oplus A_2$.
- $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$ - semiorthogonal decomposition of $\mathcal{T}$ if
  1) $\mathrm{Hom}_{\mathcal{T}}(A_2, A_1) = 0$ $\forall A_1 \in \mathcal{T}_1, A_2 \in \mathcal{T}_2$
  2) $\forall A \in \mathcal{T}$ $\exists$ dist. triangle $A_2 \to A \to A_1 \to A_2[1]$. (*)
     s.t. $A_1 \in \mathcal{T}_1, A_2 \in \mathcal{T}_2$.

Lm. triangle (*) is functorial, i.e. $\forall f: A \to B$ $\exists! f_1, f_2$:
$$\begin{tikzcd} A_2 \arrow[r, "u"] \arrow[d, dashed, "f_2"] & A \arrow[r, "v"] \arrow[d, "f"] & A_1 \arrow[r, "w"] \arrow[d, dashed, "f_1"] & A_2[1] \arrow[d, dashed, "f_2[1]"] \\ B_2 \arrow[r, "u'"'] & B \arrow[r, "v'"'] & B_1 \arrow[r, "w'"'] & B_2[1] \end{tikzcd}$$

Pf. Apply $\mathrm{Hom}_{\mathcal{T}}(-, B_1)$:
$$\underset{0}{\underset{\parallel}{\mathrm{Hom}(A_2[1], B_1)}} \to \mathrm{Hom}(A_1, B_1) \xrightarrow[\sim]{-\circ v} \mathrm{Hom}(A, B_1) \to \underset{0}{\underset{\parallel}{\mathrm{Hom}(A_2, B_1)}} \Rightarrow \exists! f_1$$
$$f_1 \longmapsto v' \circ f$$

For f_2 apply $\mathrm{Hom}(A_2, \cancel{B_2})$.

Cor 1) $\forall A \in \mathcal{T}$ dist. triangle (*) is unique up to a unique iso

2) Inclusion $i_1: \mathcal{T}_1 \hookrightarrow \mathcal{T}$ has a left adjoint i_1^*
$i_2: \mathcal{T}_2 \hookrightarrow \mathcal{T}$ has a right adjoint $i_2^!$

Pf. 1) Apply lemma to $f = id_A$

2) $i_1^* A := A_1$, $i_1^*(f) := f_1$; similarly for i_2. Exercise: check adjunctness.

<u>Rk</u> (Exercise) • $\tau = \langle \tau_1, \tau_2 \rangle \Rightarrow \tau_1 = \tau_2^{\perp} := \{A \in \tau \text{ s.t. } \mathrm{Hom}(A_2, A) = 0 \ \forall A_2 \in \tau_2\} \subseteq \tau$

$\tau_2 = {}^{\perp}\tau_1 := \{A \in \tau \text{ s.t. } \mathrm{Hom}(A, A_1) = 0 \ \forall A_1 \in \tau_1\} \subseteq \tau.$

- Conversely, if $i : \tau_1 \hookrightarrow \tau$ is an exact fully faithful functor of triang. cats. admitting a left adjoint then $\tau = \langle i(\tau_1), {}^{\perp}i(\tau_1) \rangle$

<u>Def</u> $\tau_1, \dots \tau_n \subseteq \tau$ - strictly full triang subcats

$\tau = \langle \tau_1, \dots \tau_n \rangle$ - <u>semiorth. decomposition</u> if

1) $\mathrm{Hom}(A_i, A_j) = 0 \quad \forall A_i \in \tau_i, A_j \in \tau_j, \ i > j$

2) $\forall A \in \tau \ \exists$ filtration $0 = A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \dots \to A_0 = A$ s.t.

$C(f_i) \in \tau_i$, where $A_i \xrightarrow{f_i} A_{i-1} \to C(f_i) \to A_i[1]$ - dist. triangle.

<u>Exercise</u> $n = 2$ equivalent to previous definition.

<u>Def</u> τ - $\mathbb{C}$-linear triang. cat.

- $E \in \tau$ is <u>exceptional</u> if $\mathrm{Hom}(E, E[n]) = \begin{cases} \mathbb{C}, & n = 0 \\ 0, & n \neq 0 \end{cases}$
- $(E_1, \dots E_n)$ - <u>exceptional collection</u> if $E_i \in \tau$ - exceptional $\forall i$ and $\mathrm{Hom}(E_i, E_j[n]) = 0 \ \forall i > j, n \in \mathbb{Z}$.
- exceptional collection $(E_1, \dots E_n)$ is <u>strong</u> if $\mathrm{Hom}(E_i, E_j[n]) = 0 \ \forall n \neq 0$
- exceptional collection $(E_1, \dots E_n)$ is <u>full</u> if $E_1, \dots E_n \in \tau' \subseteq \tau \Rightarrow \tau' = \tau$

strictly full triang. subcat.

<u>Def.</u> τ - $\mathbb{C}$-linear. τ is of <u>finite type</u> if $\forall A, B \in \tau \quad \bigoplus_{n \in \mathbb{Z}} \dim \mathrm{Hom}(A, B[n]) < \infty$

<u>Prop.</u> τ - $\mathbb{C}$-linear of fin. type, $E \in \tau$ - exceptional. Let $F_E : D^b(\mathbb{C}) \to \tau$ be given by $V[n] \mapsto \underset{E^{\oplus \dim V}[n].}{\overset{\cong}{V \otimes E[n]}}$ $\Rightarrow \tau = \langle \underset{D^b(\mathbb{C})}{\overset{\cong}{F_E(D^b(\mathbb{C}))}}, E^{\perp} \rangle$

$E^{\perp} = \{A \in \tau \mid \mathrm{Hom}(\underbrace{E[n], A}) = 0 \ \forall n \in \mathbb{Z}\} \subseteq \tau$

<u>Pf.</u> $\forall A$ consider dist. triangle

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(E, A[n]) \otimes E[-n] \to A \to C(ev) \to ??[1]$$

consider $\mathrm{Hom}(E[-n], -) \Rightarrow [E[-n], C(ev)] = 0 \ \forall n \in \mathbb{Z} \Rightarrow C(ev) \in E^{\perp}$ □

<u>Cor.</u> τ - $\mathbb{C}$-linear of fin. type, $(E_1, \dots E_n)$ - exc. collection $\Rightarrow$

$\Rightarrow \tau = \langle \mathcal{D}, F_{E_1}(D^b(\mathbb{C})), F_{E_2}(D^b(\mathbb{C})), \dots, F_{E_n}(D^b(\mathbb{C})) \rangle$, where $\mathcal{D} = \langle E_1, \dots E_n \rangle^{\perp}$

$= \{A \in \tau \mid \mathrm{Hom}(E_i, A[n]) = 0 \ \forall n\}$

If $(E_1, \dots E_n)$ is full then $\tau = \langle F_{E_1}(D^b(\mathbb{C})), \dots, F_{E_n}(D^b(\mathbb{C})) \rangle$ is a semiorth. decomp.

Ex: 1) $R = \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix} \leq M_3(\mathbb{C})$, (P_2, P_3, S_2) - full exceptional collection. in $D^b_f(R)$

2) $R = \mathbb{C}\langle e_1, e_2, x_1, \dots, x_n\rangle \Big/ \left(\begin{array}{l} e_2 x_i = x_i = x_i e_2 \\ e_1^2 = e_1, e_2^2 = e_2, e_1e_2 = e_2e_1 = x_ie_1 = e_2x_i = x_ix_j = 0 \end{array} \right)$

$\rightsquigarrow$ $S_1 = \mathbb{C}$, where e_1 acts as identity and the rest as 0 $\simeq R\langle x_i, e_2\rangle$

$S_2 = \mathbb{C}$, —〃— e_2. —〃— $\simeq R/\langle x_i, e_1\rangle$

$\langle S_2, S_1\rangle$ - ~~full~~ exceptional collection in $D^b_{fg}(R)$

Thm. $\mathcal{T}$ - $\mathbb{C}$-linear of fin. type, $(E_1, \dots E_n)$ - full strong exceptional collection

$\Rightarrow \mathcal{T} \simeq D^b_f(R)$, $R = (\mathrm{End}(E_1 \oplus \dots \oplus E_n))^{op}$

Def. X - top. space. A presheaf of abelian groups (rings) is a data of abelian groups (rings) $\mathcal{F}(U)$ [$\Gamma(U, \mathcal{F})$] $\forall U \underset{open}{\subseteq} X$ ("sections of $\mathcal{F}$ over U") and homomorphisms $\varphi_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$ $\forall V \underset{op}{\subseteq} U \underset{op}{\subseteq} X$ ("restriction homomorphisms", $\varphi_{UV}(s) =: s|_V$), s.t.

1) $\mathcal{F}(\emptyset) = \{0\}$ 2) $\varphi_{UU} = id_{\mathcal{F}(U)}$ 3) $\varphi_{VW} \circ \varphi_{UV} = \varphi_{UW}$ $\forall W \underset{op}{\subseteq} V \underset{op}{\subseteq} U \underset{op}{\subseteq} X$.

A presheaf $\mathcal{F}$ is a sheaf if $\forall U = \bigcup_{i \in I} V_i \subseteq X$, $V_i \underset{op}{\subseteq} X$,

4) $s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = 0$ $\forall i \Rightarrow s = 0$

5) $\{s_i \in \mathcal{F}(V_i)\}_{i \in I}$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} \Rightarrow \exists s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$

Ex: X - top space, $U \mapsto C(U, \mathbb{R})$ - sheaf of continuous real-valued functions

Def. A morphism of sheaves $f: \mathcal{F} \to G$ is a family of homomorphisms $f_U: \mathcal{F}(U) \to G(U)$ $\forall U \underset{op}{\subseteq}$ commuting with restrictions.

Def $\mathcal{F}$ - presheaf on X, $x \in X$. $\rightsquigarrow$ $\mathcal{F}_x := \varinjlim_{x \in U \underset{op}{\subseteq} X} \mathcal{F}(U)$ - stalk of $\mathcal{F}$ at x.

Thm. X - top. space. $\Rightarrow$ $\mathrm{Sh}\mathcal{A}b(X)$ is an abelian category with enough injectives

Rk. $f: \mathcal{F} \to G$ - hom-sm of sheaves. The "image" $U \mapsto f_U(\mathcal{F}(U)) \subseteq G(U)$ may fail to be a sheaf, and one needs to sheafify it to get the image in the category of sheaves.

functor left adjoint to the inclusion $\mathrm{PreSh}\mathcal{A}b \hookrightarrow \mathrm{Sh}\mathcal{A}b$, it keeps the stalks the same

In particular,

1) f is inj. $\Leftrightarrow$ $\mathcal{F}_x \hookrightarrow G_x$ is inj $\forall x \in X$

2) $\mathrm{Im}\, f = G \Leftrightarrow \mathcal{F}_x \to G_x$ is surj. $\forall x \in X$

Then f is a surj. map of sheaves

<u>Def</u> $(X, \mathcal{O}_X)$, where X is a top. space and $\mathcal{O}_X$ is a sheaf of rings is a <u>locally</u> <u>ringed</u> <u>space</u> if $\forall x \in X$ $(\mathcal{O}_X)_x$ is a local ring [i.e. has a unique max. ideal]

<u>Morphism</u> $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a cont. map $X \to Y$ and a morphism of sheaves $f_\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$, where $f_* \mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$, s.t. $\forall x \in X$ the induced homomorpsm $(\mathcal{O}_Y)_{f(x)} \to (\mathcal{O}_X)_x$ maps the maximal ideal into the max. ideal.

<u>Ex</u>: R - commutative ring $\rightsquigarrow$ $\operatorname{Spec} R := \{P \trianglelefteq R \mid P\text{-prime ideal}\}$, $\forall I \trianglelefteq R$ $V(I) := \{P \in \operatorname{Spec} R \mid P \supseteq I\} \subseteq \operatorname{Spec} R$ - closed subsets. This defines <u>Zariski</u> topology on $\operatorname{Spec} R$. Closed pts correspond to max. ideals.

$f \in R \rightsquigarrow U_f := \operatorname{Spec} R \setminus V((f))$ - <u>principal</u> <u>open</u> subsets, Base of topology.

$$U = \operatorname{Spec} R \setminus V(I) \rightsquigarrow \mathcal{O}(U) := \begin{cases} \{f \in \underset{\text{field of fractions}}{Q(R)} \mid \forall g \in I\ \exists h \in R \text{ s.t. } f = \frac{h}{g}\} & \text{- if } R \text{ is a domain} \\ \varprojlim_{U_f \subseteq U} R[f^{-1}] \end{cases}$$

<u>Def</u> $(X, \mathcal{O}_X)$ - loc. ringed space.
$(X, \mathcal{O}_X)$ is a <u>scheme</u> if $\forall x \in X$
$\exists x \in \underset{\text{open}}{U \subseteq X}$ s.t. $(U, \mathcal{O}_X|_U) \simeq (\operatorname{Spec} R, \mathcal{O})$

I.e. schemes are glued from $(\operatorname{Spec} R, \mathcal{O})$ - "affine schemes".

<u>Ex</u>. 1) $R = \mathbb{C}[x_1, \dots, x_n] \rightsquigarrow \mathbb{A}^n_{\mathbb{C}} = (\operatorname{Spec} R, \mathcal{O})$ - <u>affine</u> <u>space</u>, max. ideals $\sim$ pts in $\mathbb{C}^n$
$(x_1 - a_1, \dots, x_n - a_n) \leftrightarrow (a_1, \dots a_n) \in \mathbb{C}^n$

2) $R = \mathbb{C}[x_1, \dots x_n]/I \rightsquigarrow \operatorname{Spec} R \sim$ subset of $\mathbb{C}^n$ given by $\bigcap_{f \in I} \{f = 0\}$
prime ideals $\leftrightarrow$ irreducible subsets of $\mathbb{C}^n$ cut by polyn. equations.

3)
$$\begin{tikzcd} \mathbb{A}^1_{\mathbb{C}} - \{0\} \arrow[r, hook, "x \to x"] \arrow[d, "\wr", "x \to x^{-1}"'] & \mathbb{A}^1_{\mathbb{C}} \arrow[d] \\ \mathbb{A}^1_{\mathbb{C}} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

<u>projective</u> <u>line</u>. Maximal points: $\mathbb{P}^1_{\mathbb{C}} \simeq \{(x_1, x_2) \in \mathbb{C}^2 \setminus \{0\}\}/(x_1, x_2) \sim (\lambda x_1, \lambda x_2)$
$= \{(x_1, x_2) \in \mathbb{C}^2 \mid x_2 \neq 0\}/_\sim \cup \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 \neq 0\}/_\sim$
$= \underset{x_1/x_2}{\mathbb{C}} \cup \underset{x_2/x_1}{\mathbb{C}} \simeq \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$

Similarly, $\mathbb{P}^n_{\mathbb{C}} \simeq \{(x_1, \dots x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}\}/(x_1, \dots x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1})$

$\qquad \shortparallel$

$\mathbb{A}^n_{\mathbb{C}} \cup \dots \cup \mathbb{A}^n_{\mathbb{C}}$ $\nwarrow$ <u>projective</u> <u>space</u>

$\mathbb{A}^{n-1}_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}} - \{0\}$ $\nearrow$

Class of $(x_1, \dots, x_{n+1})$ in $\mathbb{P}^n_{\mathbb{C}}$ is denoted $[x_1 : \dots : x_{n+1}] \leftrightarrow$ line in $\mathbb{C}^{n+1}$ through $(x_1, \dots x_{n+1})$ and 0.

4) $f_1, \dots, f_m \in \mathbb{C}[X_1, \dots, X_{n+1}]_h$ $\leftarrow$ homogeneous
$\rightsquigarrow V(f_1, \dots, f_m) \underset{\text{closed}}{\subseteq} \mathbb{P}^n_{\mathbb{C}}$ given by

$$V(f_1, \dots, f_m) \cap \underset{x_i = 1 \text{ in } \mathbb{P}^n_{\mathbb{C}}}{\mathbb{A}^n_{\mathbb{C}}} := \operatorname{Spec} \mathbb{C}[x_1, \dots x_{i-1}, 1, x_{i+1}, \dots x_{n+1}] \Big/ \begin{pmatrix} f_1(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots x_{n+1}), \\ \dots \\ f_m(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots x_{n+1}) \end{pmatrix}$$

- a scheme. Its maximal pts are $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n_{\mathbb{C}}$ s.t.
$f_i(x_1, \dots x_{n+1}) = 0 \ \forall i$

Def A <u>projective</u> scheme <u>over</u> $\mathbb{C}$ is a scheme isomorphic to 4) above
A proj. scheme X over $\mathbb{C}$ is a <u>proj. variety</u> <u>over</u> $\mathbb{C}$ if $\forall x\in X$ $(\mathcal{O}_X)_x$ is a domain.

Def. $(X,\mathcal{O}_X)$-scheme. A <u>sheaf of $\mathcal{O}_X$-modules</u> is a sheaf $\mathcal{F}$ of abelian groups with $\mathcal{F}(U)$ equipped with the structure of an $\mathcal{O}_X(U)$-module $\forall U \underset{\text{open}}{\subseteq} X$ s.t. restrictions are compatible with the module structure.

Ex. R-comm. ring, $M\in R\text{-Mod} \rightsquigarrow \underset{\operatorname{Spec} R}{\overset{\cap}{U}} \mapsto \tilde{M}(U) := \varprojlim_{U_f\subseteq U} M[f^{-1}]$, where $M[f^{-1}] := \varinjlim (M \xrightarrow{f} M \xrightarrow{f} M \to \dots)$

Def. $(X,\mathcal{O}_X)$-scheme. $\mathcal{O}_X$-module $\mathcal{F}$ is
1) <u>quasi-coherent</u> if $\forall x\in X$ $\exists \underset{x}{U}\subseteq X$ s.t. $(U,\mathcal{O}_X|_U)\simeq \operatorname{Spec} R$ & $\exists M\in R\text{-Mod}$ s.t. $\mathcal{F}|_U \simeq \tilde{M}$
2) <u>coherent</u> if 1) and one can choose M to be fin. gen. (and fin. presented?)

Rk. $(X,\mathcal{O}_X)=\operatorname{Spec} R \rightsquigarrow \operatorname{QCoh}(X)\simeq R\text{-Mod}$; $\operatorname{Coh}(X)\simeq$ $\{$fin. gen. R-Mod$\}$.

Thm X-scheme $\Rightarrow \operatorname{QCoh}(X)$ is an abelian cat. with enough injectives.
X – noetherian scheme $\Rightarrow \operatorname{Coh}(X)$-abelian cat.
$X=\bigcup_{i=1}^{n} U_i$, $U_i\simeq \operatorname{Spec} R_i$, R_i-noeth. rings.

Rk. X-proj. scheme over $\mathbb{C} \Rightarrow X$-noetherian.

Rk. $f:\mathcal{F}\hookrightarrow\mathcal{G}$ – inj. hom-sm of ~~sheaves~~ QCoh sheaves. $U\mapsto \mathcal{F}(U)/\mathcal{G}(U)$ may fail to be a sheaf, one has to sheafify.

Def $D^*(\operatorname{QCoh} X)$ – derived cat. of quasi-coh. sheaves on X.
$D^b(X) := D^b(\operatorname{Coh} X)$ – derived cat. of X.

Thm. X-noetherian $\Rightarrow D^b(X)\hookrightarrow D(\operatorname{QCoh}(X))$ is a fully faithful embedding, $D^b(X)$ is equiv. to $D^b_{coh}(\operatorname{QCoh}(X))$ of complexes with bounded and coherent cohomology.

Def. X-scheme, $\mathcal{F}$-coh. sheaf on X. M is <u>locally free</u> if $\forall x\in X$ $\exists \underset{\varphi}{x\in U}\subseteq X$ s.t. $\mathcal{F}|_U \simeq \mathcal{O}_X^{\oplus n}|_U$

Rk. There is an equiv. $\left\{\begin{array}{c}\text{locally free sheaves}\\ \text{on } X \text{ of rank } n\end{array}\right\} \simeq \left\{\begin{array}{c}\text{rank } n \text{ vector}\\ \text{bundles over } X\end{array}\right\}$ $\varphi: Y\to X$, ~~s.t.~~ $\forall x\in X$ $\exists x\in U \underset{\text{open}}{\subseteq} X$ &

$$\begin{tikzcd} \mathbb{A}^n\times U \arrow[r, "\varphi_U", "\sim"'] \arrow[d, "\pi_U"'] & \varphi^{-1}(U) \arrow[dl, "\varphi"] \\ U & \end{tikzcd}$$

s.t.

$$\begin{tikzcd} \mathbb{A}^n_X(U\cap V) \arrow[r, "\varphi_U|_{U\cap V}", "\sim"'] & \varphi^{-1}(U\cap V) \arrow[d] \\ GL_n(\mathcal{O}_X(U\cap V)) \circlearrowright \mathbb{A}^n_X(U\cap V) \arrow[u, "g"] & \mathcal{O}_X|_{U\cap V} \end{tikzcd}$$

Ex. $\underset{\mathbb{P}^n_{\mathbb{C}}}{\overset{\downarrow}{\mathcal{O}(-1)}} ::= \{(\ell,v)\in\mathbb{P}^n_{\mathbb{C}}\times\mathbb{A}^{n+1}_{\mathbb{C}} \mid v\in\ell\}$ – <u>tautological line bundle over $\mathbb{P}^n$</u>

$\Gamma(U_f,\mathcal{O}(-1)) = \{\frac{p}{f^m}\in\mathbb{C}(x_1,\dots x_{n+1}) \mid p\text{-homogen}, \deg p = m-1\}$
$U_f = \mathbb{P}^n_{\mathbb{C}}\setminus V(f)\subseteq\mathbb{P}^n_{\mathbb{C}}$, f-homogen.
$\Gamma(U,\mathcal{O}(-1)) = \varprojlim_{U_f\subseteq U}\Gamma(U_f,\mathcal{O}(-1))$

$\mathcal{O}(1) := (\mathcal{O}(-1))^{\vee}$, $\mathcal{O}(\ell) := \overbrace{\mathcal{O}(1)\otimes\dots\otimes\mathcal{O}(1)}^{\ell}$, $\ell\in\mathbb{N}$; $\mathcal{O}(-\ell) := \mathcal{O}(\ell)^{\vee}$, $\ell\in\mathbb{N}$, $\mathcal{O}(0) := \mathcal{O}_{\mathbb{P}^n}$ – line bundles

$\Gamma(U_f,\mathcal{O}(\ell)) = \{\frac{p}{f^m} \mid p\text{-homogen}, \deg p = m+\ell\}$, $\Gamma(U,\mathcal{O}(\ell)) = \varprojlim_{U_f\subseteq U}\Gamma(U_f,\mathcal{O}(\ell))$.

Exercise: $\Gamma(\underset{\overset{\nearrow}{x_0=1 \text{ in } \mathbb{P}^n_{\mathbb{C}}}}{\mathbb{A}^n_{\mathbb{C}}}, \mathcal{O}(\ell)) \simeq x_i^{\ell} \cdot \mathbb{C}[\frac{x_1}{x_i}, \dots, \frac{x_{n+1}}{x_i}]$

Def R-commutative $\mathbb{C}$-algebra. $\Omega_{R/\mathbb{C}} := \bigoplus_{f \in R} R \cdot df \Big/ \left\langle \begin{matrix} dr = 0, r \in \mathbb{C} \\ d(f+g) = df + dg \\ d(fg) = fdg + gdf \end{matrix} \right\rangle$
- module of Kähler differentials

Ex: $\Omega_{\mathbb{C}[x_1,..x_n]/\mathbb{C}} \simeq \bigoplus_{i=1}^{n} \mathbb{C}[x_1,..x_n] dx_i$

Exercise: R-com. $\mathbb{C}$-algebra, $S = R[1/f_i]$ for some $\{f_i\} \subseteq R \Rightarrow \Omega_{S/\mathbb{C}} \simeq S \otimes_R \Omega_{R/\mathbb{C}}$

Def X-variety over $\mathbb{C}$. $\Omega_{X/\mathbb{C}}$ - sheaf of Kähler differentials given by $U \mapsto \lim_{\operatorname{Spec} R \subseteq U} \Omega_{R/\mathbb{C}}$. X-noetherian $\Rightarrow \Omega_{X/\mathbb{C}}$ is a coherent sheaf.

X - smooth if $\Omega_{X/\mathbb{C}}$ is a locally free sheaf.

Ex: $\mathbb{A}^n_{\mathbb{C}}$ is smooth, $\mathbb{P}^n_{\mathbb{C}}$ is smooth.

Def X-scheme $\rightsquigarrow \Gamma(X,-): QCoh(X) \to \mathcal{O}_X(X)\text{-Mod}$ is left exact

$\Rightarrow R\Gamma(X,-): D^+(QCoh(X)) \to D^+(\mathcal{O}_X(X))\text{-Mod}$.

For $\mathcal{F} \in QCoh(X)$ put $H^i(X,\mathcal{F}) := R^i\Gamma(X,\mathcal{F}) = H^i(R\Gamma(X,\mathcal{F}[0])) =$

sheaf cohomology $= H^i(Q^0(X) \to Q^1(X) \to \dots)$, where $Q^i \in QCoh$ - injective and

$0 \to \mathcal{F} \to Q^0 \to Q^1 \to \dots$ - exact, i.e.

$0 \to \mathcal{F}_x \to Q^0_x \to Q^1_x \to \dots$ is exact $\forall x \in X$.

Similarly, for $\mathcal{F}^\bullet \in D^+(QCoh(X))$, $\mathbb{H}^i(X,\mathcal{F}^\bullet) := H^i(R\Gamma(X,\mathcal{F}^\bullet))$ - hypercohomology of $\mathcal{F}^\bullet$

Thm. 1) X-scheme, $\mathcal{F} \in QCoh(X) \Rightarrow H^i(X,\mathcal{F}) \simeq \operatorname{Hom}_{D(QCoh(X))}(\mathcal{O}_X[0], \mathcal{F}[0][i])$

2) Mayer-Vietoris property: $X = U_1 \cup U_2$, $\mathcal{F} \in QCoh(X) \Rightarrow \exists$ long exact sequence

$$\dots \to H^i(X,\mathcal{F}) \to H^i(U_1,\mathcal{F}|_{U_1}) \oplus H^i(U_2,\mathcal{F}|_{U_2}) \to H^i(U_1 \cap U_2, \mathcal{F}|_{U_1 \cap U_2}) \to H^{i+1}(X,\mathcal{F}) \to \dots$$

3) $X = \bigcup_{j=1}^{n} U_j$, U_j-affine, $\mathcal{F} \in QCoh(X) \Rightarrow H^i(X,\mathcal{F}) \simeq \check{H}^i(\mathcal{U},\mathcal{F})$, where

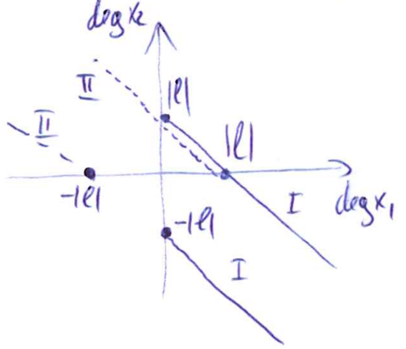
$$\check{H}^i(\mathcal{U},\mathcal{F}) := H^i\Big(\bigoplus_{j_1=1}^{n} \mathcal{F}(U_{j_1}) \to \bigoplus_{1 \le j_1 < j_2 \le n} \mathcal{F}(U_{j_1} \cap U_{j_2}) \to \dots\Big)$$

$$\mathcal{F}(U_{j_1} \cap \dots \cap \widehat{U_{j_k}} \cap \dots \cap U_{j_m}) \to \mathcal{F}(U_{j_1} \cap \dots \cap U_{j_m})$$
$$s \longmapsto (-1)^{k+1} \cdot s|_{U_{j_1} \cap \dots \cap U_{j_m}}.$$

Rk. X-smooth proj. variety $/\mathbb{C} \Rightarrow H^i_{\text{sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \bigoplus_{p+q=i} H^p(X, \Lambda^q \Omega_{X/\mathbb{C}})$ - Hodge decomposition.

singular cohomology with $\mathbb{C}$-coefs. of the set of max. pts of X in $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^* \simeq \mathbb{C}P^n$ with Euclidean topology

Ex: $H^i(\mathbb{P}^1_\mathbb{C}, \mathcal{O}(\ell)) \simeq H^i\big(\mathcal{O}(\ell)(\mathbb{A}^1) \oplus \mathcal{O}(\ell)(\mathbb{A}^1) \xrightarrow{f\oplus g \ \mapsto\ f-g} \mathcal{O}(\ell)(\mathbb{A}^1 - \{0\})\big)$



$$\mathcal{O}(\ell)(\mathbb{A}^1) \oplus \mathcal{O}(\ell)(\mathbb{A}^1) = x_2^\ell \cdot \mathbb{C}[x_1/x_2] \oplus x_1^\ell \cdot \mathbb{C}[x_2/x_1] = \bigoplus_{n\geq 0} \mathbb{C}\cdot \frac{x_1^n}{x_2^{n-\ell}} \oplus \bigoplus_{n\geq 0} \mathbb{C}\cdot \frac{x_2^n}{x_1^{n-\ell}}$$

$$\mathcal{O}(\ell)(\mathbb{A}^1 - \{0\}) = x_2^\ell \cdot \mathbb{C}[x_1/x_2, x_2/x_1] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot \frac{x_1^{n+\ell}}{x_2^n}$$

$\Rightarrow$ $\dim_\mathbb{C} H^0(\mathbb{P}^1_\mathbb{C}, \mathcal{O}(\ell)) = \begin{cases} 0, & \ell < 0 \\ \ell+1, & \ell \geq 0 \end{cases}$

$\dim_\mathbb{C} H^1(\mathbb{P}^1_\mathbb{C}, \mathcal{O}(\ell)) = \begin{cases} 0, & \ell \geq 0 \\ -\ell-1, & \ell < 0 \end{cases}$

Exercise $\dim_\mathbb{C} H^i(\mathbb{P}^n_\mathbb{C}, \mathcal{O}(\ell)) = \begin{cases} \binom{\ell+n}{n}, & i=0, \ell \geq 0 \\ \binom{-\ell-1}{n}, & i=n, \ell \leq -n-1 \\ 0, & \text{otherwise} \end{cases}$

Prop $(\mathcal{O}_{\mathbb{P}^n_\mathbb{C}}(-n)[0], \mathcal{O}_{\mathbb{P}^n_\mathbb{C}}(-n+1)[0], \dots, \mathcal{O}_{\mathbb{P}^n_\mathbb{C}}[0])$ – strong exceptional collection in $D^b(\mathbb{P}^n_\mathbb{C})$.

Pf $\mathrm{Hom}_{D^b(\mathbb{P}^n_\mathbb{C})}(\mathcal{O}(\ell)[0], \mathcal{O}(m)[i]) \simeq \mathrm{Hom}_{D^b(\mathbb{P}^n_\mathbb{C})}(\mathcal{O}_{\mathbb{P}^n_\mathbb{C}}[0], \mathcal{O}_{\mathbb{P}^n_\mathbb{C}}(m-\ell)[0][i]) \simeq H^i(\mathbb{P}^n_\mathbb{C}, \mathcal{O}(m-\ell))$

$- \overset{L}{\otimes} \mathcal{O}(-\ell)$ is invertible on $D^b(\mathbb{P}^n_\mathbb{C})$ with the inverse $- \overset{L}{\otimes} \mathcal{O}(\ell)$

$= \begin{cases} \mathbb{C}^{\binom{m-\ell+n}{n}}, & m \geq \ell, i=0 \\ 0, & \text{otherwise, since } -n \leq m-\ell \end{cases}$

Thm X - proj. var / $\mathbb{C}$, $\mathcal{F} \in \mathrm{Coh}(X) \Rightarrow \dim_\mathbb{C} H^i(X, \mathcal{F}) < \infty$ $\forall i$

(2) If X is smooth, then $D^b(X)$ is of finite type.

Def. $f: Y \to X$ - morphism of varieties over $\mathbb{C}$.

$f_*: \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ – direct image

$\mathcal{F} \mapsto f_*\mathcal{F}$, $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$ - module over $\mathcal{O}_Y(f^{-1}(U)) \Rightarrow$ over $\mathcal{O}_X(U)$

f_* is left exact and has right adjoint

$f^*: \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.

f^* is right exact

Ex: $f: Y \to X$, $Y = \mathrm{Spec}\, R$, $X = \mathrm{Spec}\, S$ $\rightsquigarrow$ f corresponds to $f_\#: S \to R$ – hom-sm.

$$\begin{tikzcd} \mathrm{QCoh}(Y) \arrow[r, "f_*"] \arrow[d, phantom, "\wr"] & \mathrm{QCoh}(X) \arrow[d, phantom, "\wr"] \\ R\text{-Mod} \arrow[r, "\text{forgetful}"] & S\text{-Mod} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "f^*"] \arrow[d, phantom, "\wr"] & \mathrm{QCoh}(Y) \arrow[d, phantom, "\wr"] \\ S\text{-Mod} \arrow[r, "R\otimes -"] & R\text{-Mod} \end{tikzcd}$$

Thm $f: Y \to X$ – morphism of projective varieties over $\mathbb{C}$. $\rightsquigarrow$ $Rf_*: D^b(Y) \to D^b(X)$

$Rf^*: D^b(X) \to D^b(Y)$

Ex: $f: X \to \mathrm{Spec}\,\mathbb{C} \rightsquigarrow f_*: \mathrm{QCoh}(X) \to \mathbb{C}\text{-Vect}$.

$\mathcal{F} \mapsto \mathcal{F}(X) = \Gamma(X, \mathcal{F})$

$$\begin{tikzcd} D(\mathrm{QCoh}(X)) \arrow[r, "Rf_*"] & D(\mathbb{C}\text{-Vect}) \\ D^b(X) \arrow[u, phantom, "\cup"] \arrow[r] & D^b_f(\mathbb{C}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

in particular, $R^if_*(\mathcal{F}) \simeq H^i(X, \mathcal{F})$

Properties:

1) $R(f\circ g)_* = Rf_* \circ Rg_*$, $L(f\circ g)^* = Lf^* \circ Lg^*$, $X \xrightarrow{g} Y \xrightarrow{f} Z$

2) $\underset{\text{proj.}}{f: Y \to X}$, $\mathcal{F} \in D^b(Y)$, $G \in D^b(X) \rightsquigarrow \mathrm{Hom}_{D^b(Y)}(Lf^*G, \mathcal{F}) \simeq \mathrm{Hom}_{D^b(X)}(G, Rf_*\mathcal{F})$

Rk. $M^\bullet, N^\bullet \in \mathrm{Kom}_\bullet(\mathrm{QCoh}(X)) \rightsquigarrow M^\bullet \otimes N^\bullet \in \mathrm{Kom}_\bullet(\mathrm{QCoh}(X))$, $(M^\bullet \otimes N^\bullet)^n := \bigoplus_{i+j=n} M^i \otimes N^j$,
$d(\underset{M^i \otimes N^j}{\underset{\uparrow}{m \otimes n}}) = (dm)\otimes n + (-1)^i m \otimes dn \rightsquigarrow$ descends to $\overset{L}{\otimes}$ on ~~[illegible]~~ $D^b(X)$ for X-smooth; $-\overset{L}{\otimes}-$ is exact in both variables, e.g. $M_1^\bullet \to M_2^\bullet \to M_3^\bullet \to M_1^\bullet[1]$ - dist. triangle
$\Rightarrow M_1^\bullet \otimes N^\bullet \to M_2^\bullet \otimes N^\bullet \to M_3^\bullet \otimes N^\bullet \to M_1^\bullet \otimes N^\bullet[1]$ - dist. triangle.

3) $\mathcal{F}^\bullet, G^\bullet \in D^b(X)$, $f: Y \to X \rightsquigarrow Lf^*(\mathcal{F}^\bullet \overset{L}{\otimes} G^\bullet) \simeq Lf^*(\mathcal{F}^\bullet) \overset{L}{\otimes} Lf^*(G^\bullet)$

4) Projection formula: $f: Y \to X$ - smooth, proj. varieties / $\mathbb{C}$, $\mathcal{F}^\bullet \in D^b(Y)$, $G^\bullet \in D^b(X)$
$\Rightarrow Rf_*(\mathcal{F}^\bullet \overset{L}{\otimes} Lf^* G^\bullet) \simeq Rf_*(\mathcal{F}^\bullet) \overset{L}{\otimes} G^\bullet$

5) X, Y - sm. proj. var / $\mathbb{C}$, ~~[illegible]~~
$\mathcal{F} \in D^b(X)$

$$\begin{tikzcd} X\times Y \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$
- cart. square

$\Rightarrow Rp_* Lq^*(\mathcal{F}) \simeq Lg^* \underbrace{Rf_*(\mathcal{F})}_{D^b_f(\mathbb{C})} \leftarrow$ finite complex of ~~[illegible]~~ free $\mathcal{O}_Y$-Mod. with 0 morphisms.

Def X, Y - sm. proj - var / $\mathbb{C}$, $X \xleftarrow{q} X\times Y \xrightarrow{p} Y$ - projections, $K \in D^b(X \times Y)$.

$$\begin{aligned} \Phi_K : D^b(X) &\to D^b(Y) \\ \mathcal{F} &\mapsto Rp_*(Lq^*\mathcal{F} \overset{L}{\otimes} K) \end{aligned}$$
- Fourier-Mukai (integral) transform with kernel K.

From now on omit R, L; all functors between $D^b(-)$ are derived.

Ex: 1) $f: X \to Y$ - sm. proj. var / $\mathbb{C}$, $i: \underset{x \mapsto (x, f(x))}{X \to X\times Y}$, $K := i_* \mathcal{O}_X$

$$\begin{tikzcd} & X\times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

$\Phi_K(\mathcal{F}) = p_*(q^*\mathcal{F} \otimes i_*\mathcal{O}_X) = \underbrace{p_* i_*}_{f_*}(\underbrace{i^* q^*}_{\mathrm{Id}^*}\mathcal{F} \otimes \mathcal{O}_X) = f_*\mathcal{F}$

2) $f: Y \to X$ - sm. proj. var / $\mathbb{C}$, $i: \underset{y \mapsto (f(y), y)}{Y \to X \times Y}$, $K := i_* \mathcal{O}_Y$

Exercise:

$$\begin{tikzcd} & X\times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \arrow[ul, "i"] \arrow[ll, "f"] \end{tikzcd}$$

$\Phi_K(\mathcal{F}) = p_*(q^*\mathcal{F} \otimes i_*\mathcal{O}_Y) = \underbrace{p_* i_*}_{\mathrm{Id}_*}(\underbrace{i^* q^*}_{f^*}\mathcal{F} \otimes \mathcal{O}_Y) = f^*\mathcal{F}$

3) $\underset{x \mapsto (x,x)}{X \xrightarrow{\Delta} X \times X}$, $K := \Delta_* \mathcal{O}_X \Rightarrow \Phi_K = \mathrm{Id}: D^b(X) \to D^b(X)$

Thm (Orlov 2003) X, Y - smooth. proj. var. / $\mathbb{C}$, $F: D^b(X) \to D^b(Y)$ - equiv. of triang. cats
$\Rightarrow F \simeq \Phi_K$ for some $K \in D^b(X \times Y)$.

Rk. $R := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \leq M_3(\mathbb{C})$ $\tilde{R} := \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \leq M_3(\mathbb{C}) \Rightarrow D^b_f(\tilde{R}) \simeq D^b_f(R)$ $\sim \Phi_T$
$M^\bullet \mapsto {}_R T_{\tilde{R}} \overset{L}{\otimes} M^\bullet$, ${}_R T_{\tilde{R}} \in D^b(R \otimes \tilde{R}^{op})$

Thm (Beĭlinson '79) Let $\pi_1, \pi_2 : \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ be projections, for $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(\mathbb{P}^n_{\mathbb{C}})$ put $\mathcal{F}^{\circ} \boxtimes \mathcal{G}^{\circ} := \pi_1^* \mathcal{F} \otimes \pi_2^*(\mathcal{G})$; Let $\Delta : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ be the diagonal. Then $\exists$ exact sequence:

$$0 \to \Lambda^n(\overset{\Omega \otimes \mathcal{O}(1)}{\Omega(1)} \boxtimes \mathcal{O}(-1)) \to \Lambda^{n-1}(\Omega(1) \boxtimes \mathcal{O}(-1)) \to \dots \to \Omega(1) \overset{\text{"}\Lambda^1\text{"}}{\boxtimes} \mathcal{O}(-1) \to \overset{\text{"}\Lambda^0\text{"}}{\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}} \boxtimes \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}} \to \Delta_* \mathcal{O}_{\mathbb{P}^n} \to 0$$

"resolution of the diagonal"

Proof (sketch): $\Gamma(\mathbb{P}^n_{\mathbb{C}}, \Omega^\vee(-1)) \simeq \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}, \Omega^\vee(-1)) \simeq \mathrm{Hom}(\Omega, \mathcal{O}(-1))$.

$$\overset{\Psi}{\frac{\partial}{\partial x_i}} \longleftarrow\!\!\!\longrightarrow \overset{\Psi}{d f} \mapsto \frac{\partial f}{\partial x_i}$$ – rat. function of degree -1

(f – rat. function of degree 0)

$$\Gamma(\mathbb{P}^n \times \mathbb{P}^n, \underbrace{\Omega^\vee(-1) \boxtimes \mathcal{O}(1)}_{\mathcal{V}\,!!}) \ni \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i} \otimes y_i =: S$$

(coord. x_i, coord. y_i)

$$\rightsquigarrow \; 0 \to \Lambda^n \mathcal{V}^\vee \to \Lambda^{n-1} \mathcal{V}^\vee \to \dots \to \Lambda^2 \mathcal{V}^\vee \to \Lambda^1 \mathcal{V}^\vee \to \mathcal{O} \to 0$$

$$\Lambda^m \mathcal{V}^\vee \longrightarrow \Lambda^{m-1} \mathcal{V}^\vee$$

$$v_1 \wedge \dots \wedge v_m \mapsto \sum_{i=1}^{m} (-1)^{i+1} v_i(S) \cdot v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge v_m$$

– Koszul resolution; local computation shows that it is exact except in the last term, where $\mathcal{V}^\vee \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}} \to (i_Z)_* \mathcal{O}_Z \to 0$, where $i_Z : Z \to \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ is the embedding of $\{S = 0\} \hookrightarrow \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$

$\boxed{\text{S transversal to zero section}}$

local computation shows $\{S=0\} = \Delta(\mathbb{P}^n_{\mathbb{C}})$. locally: R-com. ring, $M \in \mathrm{Mod}\, R$, $s \in M$, $P \trianglelefteq R$

$\{S=0\} = \{P \trianglelefteq R \mid s \in P M_P \subseteq M_P\} \subseteq \mathrm{Spec}\, R$ (localization)

Exercise: $K_1 \to K_2 \to K_3 \to K_1[1]$ – dist. triangle in $D^b(X \times Y)$

$\Rightarrow \forall \mathcal{F} \in D^b(X)$ $\exists$ a natural dist. triangle $\Phi_{K_1}(\mathcal{F}) \to \Phi_{K_2}(\mathcal{F}) \to \Phi_{K_3}(\mathcal{F}) \to \Phi_{K_1}(\mathcal{F})[1]$.

Corollary $(\mathcal{O}(-n) \boxtimes \mathcal{O}, \mathcal{O}(-n+1) \boxtimes \mathcal{O}, \dots, \mathcal{O}(-1) \boxtimes \mathcal{O})$ is a strong full exceptional collection in $D^b(\mathbb{P}^n_{\mathbb{C}})$.

Pf: strong, exceptional – already proved; remains fullness.

$$\Lambda^m(\Omega(1) \boxtimes \mathcal{O}(-1)) \underset{\uparrow}{\simeq} \Omega^m(m) \boxtimes \mathcal{O}(-m)$$ $\rightsquigarrow$ split resolution of the diagonal into

$\Lambda^m(L \otimes V) \simeq L^{\otimes m} \otimes \Lambda^m V$ for L – dim 1 v.sp; V – arbitrary

$(\Lambda^m \Omega) \otimes \mathcal{O}(m)$

$$0 \to \Omega^n(n) \boxtimes \mathcal{O}(-n) \to \Omega^{n-1}(n-1) \boxtimes \mathcal{O}(-n+1) \to C_{n-1} \to 0$$

$$0 \to C_{n-1} \to \Omega^{n-2}(n-2) \boxtimes \mathcal{O}(-n+2) \to C_{n-2} \to 0$$

$$\dots$$

$$0 \to C_1 \to \mathcal{O} \boxtimes \mathcal{O} \to \Delta_* \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}} \to 0.$$

$\rightsquigarrow$ dist. triangles $\forall \mathcal{F} \in D^b(\mathbb{P}^n)$

$$\Phi_{\Omega^n(n) \boxtimes \mathcal{O}(-n)}(\mathcal{F}) \to \Phi_{\Omega^{n-1}(n-1) \boxtimes \mathcal{O}(-n+1)}(\mathcal{F}) \to \Phi_{C_{n-1}}(\mathcal{F}) \to \Phi(\mathcal{F})[1]$$

$$\dots$$

$$\Phi_{C_1}(\mathcal{F}) \to \Phi_{\mathcal{O} \boxtimes \mathcal{O}}(\mathcal{F}) \to \Phi_{\Delta_* \mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}) \to \Phi_{C_1}(\mathcal{F})[1]$$

$$\Phi_{\Omega^m(m)\boxtimes\mathcal{O}(-m)}(\mathcal{F}) = (\pi_2)_*(\pi_1^*\mathcal{F}\otimes(\Omega^m(m)\boxtimes\mathcal{O}(-m))) = (\pi_2)_*(\pi_2^*\mathcal{F}\otimes\pi_1^*\Omega^m(m)\otimes\pi_2^*\mathcal{O}(-m)) \underset{\substack{\uparrow \\ \text{projection} \\ \text{formula}}}{=}$$

$$= (\pi_2)_*(\pi_1^*(\mathcal{F}\otimes\Omega^m(m))\otimes\mathcal{O}(-m)) = \underbrace{p_1^*(p_2)_*\overset{\in D^b(\mathbb{C})}{(\mathcal{F}\otimes\Omega^m(m))}}_{\substack{\wr\wr \\ \bigoplus \mathcal{O}_{\mathbb{P}^n}^{d_i}[-i]}}\otimes\mathcal{O}(-m)$$

$$\begin{tikzcd} & \mathbb{P}^n\times\mathbb{P}^n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n \arrow[dr, "p_2"'] & & \mathbb{P}^n \arrow[dl, "p_1"] \\ & \operatorname{Spec}\mathbb{C} & \end{tikzcd}$$

$\bigoplus \mathcal{O}_{\mathbb{P}^n}^{d_i}[-i]$, where $i = \dim_{\mathbb{C}} H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}\otimes\Omega^m(m))$

$$\Rightarrow \Phi_{\Omega^m(m)\boxtimes\mathcal{O}(-m)}(\mathcal{F}) \in \overset{\text{triang. subcat. gen. by } \mathcal{O}(-m)[0]}{\langle\mathcal{O}(-m)[0]\rangle} \subseteq D^b(\mathbb{P}^n_{\mathbb{C}})$$

$$\Rightarrow \Phi_{C_m}(\mathcal{F}) \in \langle\mathcal{O}(-n)[0],\ldots,\mathcal{O}(-m)[0]\rangle \subseteq D^b(\mathbb{P}^n_{\mathbb{C}}).$$

$$\Rightarrow \mathcal{F} = \Phi_{\Delta_*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}}(\mathcal{F}) \in \langle\mathcal{O}(-n)[0],\ldots,\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}[0]\rangle \subseteq D^b(\mathbb{P}^n_{\mathbb{C}}) \quad \square.$$

Cor. $D^b(\mathbb{P}^n_{\mathbb{C}}) \simeq D^b_f(R)$, where $R := \operatorname{End}(\mathcal{O}(-n)\oplus\ldots\oplus\mathcal{O})^{op}$ - fin. dim. non-commutative algebra$/\mathbb{C}$.

What is known in general:

Thm $D^b(X)$ admits a full exceptional collection for the following X:

1) $Gr_{\mathbb{C}}(k,n)$, Kapranov'84

2) $Q = \{\sum_{i=1}^{n+1} x_i^2 = 0\} \subseteq \mathbb{P}^n_{\mathbb{C}}$, Kapranov'88

These are projective homogeneous varieties: $\mathbb{P}^n_{\mathbb{C}} \simeq GL_{n+1}/\overset{\text{stabilizer of } [1:0:\ldots:0]}{\begin{pmatrix} * & \boxed{X} \\ 0 & \end{pmatrix}}$, $Gr_{\mathbb{C}}(k,n) \simeq GL_n/\begin{pmatrix} \overset{k}{\boxed{X}} & X \\ 0 & \boxed{X} \end{pmatrix}$, $Q \simeq SO_{n+1}/P$

$\rightsquigarrow$ some other proj. homogeneous varieties by Kuznetsov, Polishchuk, Manivel, Samokhin, Fonarev, Guseva, ...

3) $X = G/P$ - projective variety homogeneous under a split semisimple alg. group
Samokhin-van der Kallen '24

Idea: $V \in \operatorname{Rep} P \Rightarrow \underset{\substack{\downarrow \\ G/P}}{(G\times V)/P} \leftarrow (g,v)\sim(gh,h^{-1}v)$ - vector bundle over G/P

$\rightsquigarrow D^b(\operatorname{Vect}(P)) \to D^b(G/P) \rightsquigarrow$ study $\operatorname{Rep} P$ over $\operatorname{Rep} G \rightsquigarrow$ explicit construction of exceptional collection from $\operatorname{Rep} P$

4) X-smooth projective toric variety, Kawamata'18

5) Some Fano varieties

<u>Conjecture</u> $D^b(X)$ has a full exceptional collection $\Rightarrow X$ is rational
(i.e. $\exists U \subseteq X$, $W \subseteq \mathbb{A}^n$, $U \simeq W$).

How good invariant $D^b(X)$ is?

<u>Thm</u> (Bondal-Orlov'02) X, Y - sm. proj. var$/\mathbb{C}$, ω_X or $\omega_X^\vee$ is <u>ample</u>
(roughly: $H^0(X, \omega_X^{\otimes n})$ is very big for some n)

$$D^b(X) \simeq D^b(Y) \Rightarrow X \simeq Y.$$

<u>Thm</u> (Anel-Toën'09) X-sm. proj. var$/\mathbb{C} \Rightarrow \exists$ at most countable number of pairwise non-isomorphic Y s.t. $D^b(X) \simeq D^b(Y)$

Thm (Lesieutre '13) There is an infinite sequence of smooth rational projective 3-folds X_p s.t. $D^b(X_p) \simeq D^b(X_{p'})$, but $X_p \not\simeq X_{p'}$ $\forall p \neq p'$. These are some blowups of $\mathbb{P}^3$ in 8 pts.

dual abelian variety

Thm. (Mukai '81) A-abelian variety $\Rightarrow D^b(A) \simeq D^b(\hat{A})$, given by $\Phi_{\mathcal{P}}$, where

pts $\simeq \mathbb{C}^n/\Lambda$, Λ-lattice

$\mathcal{P}$ is the Poincaré line bundle on $A \times \hat{A}$

Thm (Polishchuk '02) A, B - abelian varieties. $D^b(A) \simeq D^b(B) \iff \exists f : A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$ - iso s.t. $\hat{f} = f^{-1}$

Thm. (Orlov '98) X, Y - K3-surfaces. $D^b(X) \simeq D^b(Y) \iff \tilde{H}^*(X, \mathbb{Z}) \simeq \tilde{H}^*(Y, \mathbb{Z})$

projective surface, $\omega_X \simeq \mathcal{O}_X$, not abelian variety

Hodge isometry.